On Black Holes and Hidden Dimensions

On Quantum Theory and Cayley-Dickson algebras

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Contents

Summary

This document is part of my project "Our universe is just curved spacetime". The agenda of the project is described in (PDF) Our universe is just curved spacetime Our universe is just curved spacetime (researchgate.net) . There, the rough model is described.

In the document »*On with the Big Bang*« (see [2]) we found a possible correlation between the “expansion part” and the “rotational part” of the »*dynamic Minkowski metric*« M*, that ensured that the equation together with certain transformations of G_{15} (enlarged Poincaré group with scaling factor) results in a singularity free evolution compliant to the above agenda. Although it was noted, that this correlation is just a possible one, it already showed, that the universe as a black hole would be nothing happening once, but a periodical process. So, when we try to find black holes in our universe, viewed as just curved spacetime, we should search for structures in this curved spacetime, that are periodical, too. But with periods much smaller than the one, our universe expands and shrinks again. From ordinary theory of black holes we know, that – *from an outer view on black holes (!)* - mass, charge and spin are the essential parameters of a black hole and that mass is equivalent to the Schwarzschild radius. So, do we find our just curved spacetime black holes, and especially the elementary particles among them, in certain types of black holes similar to our universe?

In this document I will argue: Yes and no. Objects, we nowadays call black holes (built up from high concentration of masses), shall be of type universe. With one exception: so-called neutrino-type black holes, that could be relicts of the very early universe, before CMB (where *CMB* denotes the phase of recombination), and which are therefore candidates for seeds of the supermassive black holes. But for elementary particles I see things differently. Roughly spoken, I will argue, that what some physicists call curled up (hidden) dimensions in an actually higher dimensional world may be nothing else then black holes - within a universe that is just curved spacetime -, that correspond to our elementary particles like electrons, neutrinos and others.

The elementary particles interact with spacetime via a free parameter $\boldsymbol{\lambda} = \boldsymbol{l}^2$ in the conformal transformations G_{15} which leave the Minkowski metric up to scaling unchanged. This will be accomplished by defining the elementary particles as so-called *roots of unity* within a Cayley-Dickson-algebra containing spacetime as the subalgebra $\mathbb{H}$ of quaternions (see section *Interaction of spacetime and elementary particles*).

In the initial phase of our universe at big bang elementary particles shall have been created in a high number of types. Types, that correspond – with respect to inner symmetries - to potency associative algebras of high dimensions (as for example the 16-dimensional algebra of *sedenion*s). Those that corresponded to algebras of dimension higher than 8, “suffered” from zero divisors in elements of the unit sphere, and *lived* only for a very short time. Only black holes with inner symmetries corresponding to real, complex numbers, quaternions and octonions (highest dimensional potency associative division algebra) survived. There is an exception: The *sedenions*. Generations I and II of elementary particles can be identified with 2 subalgebras isomorphic to the octonions. Generation III corresponds to an 8-dimensional subalgebra, that contains zero divisors. Nevertheless, it contains objects associated with so-called *roots of unity*, that avoid these zero divisors. These objects interact and change our view on the universe. The change is driven by the factor l^2 occurring in the transformation of dynamic Minkowski metric M* (for dynamic radial velocity) by transformations of G_{15}.

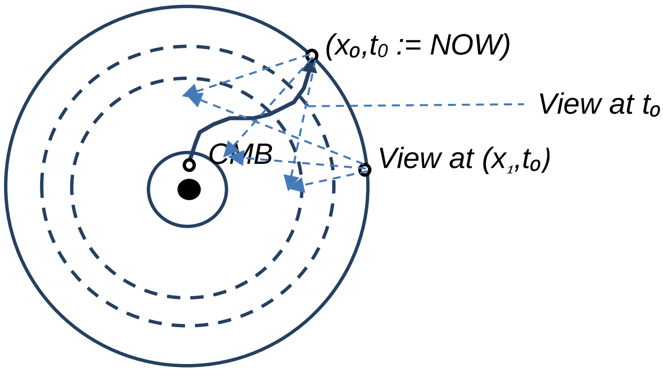
One basic idea of the project Our universe is just curved spacetime, is, that any physical object may only be defined up to an uncertainty. For elementary particles this will be the Heisenberg uncertainty principle (here derived from the mere construction of elementary particles as black holes; see section *On Heisenberg’s uncertainty principle*) and automorphisms (interpreted as interaction of particles). For spacetime itself this will be transformations of G_{15} and the interaction with elementary particles. So, it would be uncertainty that makes the world go on.

Our view on the universe

My big picture is, that our universe is just curved spacetime, and "matter" built up by extreme manifolds, the black holes. Through their encapsulation, they are the only manifolds that build up something we call objects. So, the words "masses curve spacetime" are modified to "*curved spacetime provides the illusion of masses*". This means of course, that especially all the objects, that we call elementary particles, should be black holes in curved spacetime, too.

How did our universe evolve to produce our view on it?

Our universe will be modelled as a dual construction of evolution and our view on this evolution (see [10]). For the calculations to come we use the picture on the right side.



Here *CMB* denotes the phase of recombination, and *NOW* is our time, the point in spacetime of our view on the universe. The curved line visualizes the journey of a collection of black holes in a generalized Minkowski universe (non constant radial velocity).

Now, what is creating evolution and views on this evolution, if one assumes that there is only curved spacetime and matter built up by extreme manifolds, black holes? To me it seems natural, that this must be transformations of the kind that they are scaling the Minkowski metric, written (with signature (+,-,-,-)) as:

(R) $\partial(ct')^2 - \partial x'^2 - \partial y'^2 - \partial z'^2 = \lambda(\partial(ct)^2 - \partial x^2 - \partial y^2 - \partial z^2)$, with c speed of light and time t, $\lambda = l^2 \; with \; l \in \mathbb{R}$ and (x',y',z',ct') being a transformation of (x,y,z,ct) in $\mathbb{R}^4$.

In the beginning of the last century, scientists like Poincaré and Einstein [1] evaluated these transformations and found it to be the 15-parameter group G_{15} of conformal transformations with reciprocal radii [1]. The 10-parameter Poincaré group (containing the 6-parameter Lorentz-group) is contained as a subgroup. Cunningham [1] showed, that the transformations of G_{15} can be split into three parts:

$\lambda = 1$: The Poincaré group (Lorentz-group + translations)

◎ $\lambda \in \mathbb{R}$: Affinity transformations (contains the Poincaré group)

◉ $\lambda = \frac{k^4}{((ct)^2-(x^2+y^2+z^2))^2}$: Transformations with reciprocal radii

In the document »*On with the Big Bang*« [2] we derived the »*dynamic Minkowski metric*« (v amount of radial velocity *not* constant) in spherical coordinates

$$\partial s^2 = \left(\partial(ct) - \gamma^2 \frac{\partial v}{c} r\right)^2 - \left(\partial r - \gamma^2 \frac{\partial v}{c} ct\right)^2 - \gamma^2 (r - \frac{v}{c} ct)^2 \left(\partial\theta^2 + sin^2\theta\partial\varphi^2\right) \qquad \textbf{(M*)}$$

and found it to be preserved up to scaling by transformations of G15*, meaning:

$(ct')^2 - r'^2 = \lambda((ct)^2 - r^2)$ [1] and

$\left(\partial(ct') - \gamma'^2 \frac{\partial v'}{c} r'\right)^2 - \left(\partial r' - \gamma'^2 \frac{\partial v\prime}{c} ct'\right)^2 - \gamma'^2 (r' - \frac{v'}{c} ct')^2 \left(\partial\theta'^2 + sin^2\theta'\partial\varphi'^2\right) =$

$\boldsymbol{\lambda}[\left(\partial(ct) - \gamma^2 \frac{\partial v}{c} r\right)^2 - \left(\partial r - \gamma^2 \frac{\partial v}{c} ct\right)^2 - \gamma^2 (r - \frac{v}{c} ct)^2 \left(\partial\theta^2 + sin^2\theta\partial\varphi^2\right)]$ with

◎* $\boldsymbol{\lambda} = \boldsymbol{l}^2$ for transformations of type ◎* defined by:
$r' = \gamma l (r - \frac{v}{c} ct)$, $ct' = \gamma l (ct - \frac{v}{c} r)$, θ' = θ, φ' = φ, with *v in direction of r not constant*,

$\lambda = -l^2$ for transformations of type ◎* defined by:
$r' = lct$, $ct' = lr$, θ' = θ, φ' = φ (inverts the signature of the metric)

◉ $\lambda = \frac{k^4}{((ct)^2 - r^2)^2}$ for transformations of type ◉ defined by:
$r' = \frac{k^2 r}{(ct)^2 - r^2}$, θ' = θ , φ' = φ , $ct' = \frac{k^2 ct}{(ct)^2 - r^2}$ (inversion into a hypersphere)

Transformations of type ◉ applied twice, yield the identity. We also found, that transformations of type ◉ invert the expansion behaviour.

The basic idea of what is to come will be to "blow up" $\boldsymbol{\lambda} = \boldsymbol{l}^2$ from ± 1 to "roots of unity" in an algebra of dimension 16, called the *sedenions*, and so-called summed roots of unity with a certain periodical lifecycle will be the candidates for the elementary particles of the standard model. And their interaction may take place at location $\boldsymbol{x}$ in 4-dimensional space, when they "join" the reals (time) at -1 (past) or +1 (future) in their lifecycle, according to (M*). From the standpoint of 4-dimensional spacetime, the *sedenions* share a 4-dimensional subalgebra $\mathbb{H}$ (quaternions) with spacetime. Thus 12 dimensions will be hidden, more precisely 3*4, since the *sedenions* are covered by three 8-dimensional subalgebras (generations), that all share $\mathbb{H}$, two of them isomorphic to the 8-dim. division algebra of octonions, the third one to an 8-dim. subalgebra with zero divisors to be avoided in construction of objects and their interaction. All of the elements of the sedenions share the property that a product with an element of $\mathbb{H}$ (spacetime) is only zero when one of the factors is zero.

Interaction of spacetime and elementary particles

For change of spacetime within the quaternions $\mathbb{H}$ we have with $\boldsymbol{\partial x} = \sum_{i=0}^{3} \partial x_i \boldsymbol{e}_i \in \mathbb{H}, \partial x_i \in \mathbb{R}$, $\boldsymbol{e}_i$ building a basis of $\mathbb{H}, 0 \le i \le 3$, and $\partial(ct) := \partial x_0$, using the multiplication in $\mathbb{H}$:

$$\boldsymbol{\partial x^2} = (\partial(ct)^2 - (\partial {x_1}^2 + \partial {x_2}^2 + \partial {x_3}^2))\boldsymbol{e}_0 + 2\partial(ct)(\partial x_1 \boldsymbol{e}_1 + \partial x_2 \boldsymbol{e}_2 + \partial x_3 \boldsymbol{e}_3)$$

The time coordinate ($\boldsymbol{e}_0$) just describes the ordinary Minkowski metric, the space part kind of an expansion of space. The 15-parameter group G_{15} of conformal transformations with reciprocal radii leaves the metric up to scaling unchanged. With its "scaling parameter" $\boldsymbol{\lambda} = \boldsymbol{l}^2$ it enables the definition of an interaction of certain elements of the 8-dimensional octonion algebra $\mathbb{O}$ (one of 3 subalgebras within the sedenions defining what is called

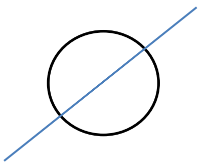
[1] For λ=1 this corresponds to conservation of angular momentum (conservation of momentum will be just the special case of conservation of angular momentum for large radius)

generations) containing ℍ to interact with spacetime. For example, we will see, that rotations of G_{15} may be defined by left-right-multiplications of certain elements (so-called roots of unity) of 𝕆 associated to quarks.

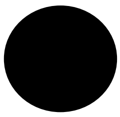
Black holes and hidden dimensions

According to the project agenda, objects in our universe shall be built from collections of black holes. What types of black holes may exist? One candidate is a type similar to our universe itself, since there is no reason, why preservation of above metrics *up to scaling* should not lead to black holes like our universe, but of smaller scale than our universe. In [2] we discussed the properties of a black hole of type universe.

In this paper we make the hypothesis, that all the black holes, that are built up by massive concentration of masses (large collections of black holes) within our universe are up to scaling the same as our universe – with the exception of so-called neutrino-type black holes (candidates for supermassive black-holes), that could be produced in the early phase of our universe. For elementary particles, the hypothesis is, that things will be different. But there is at least one thing, that all the black holes in our universe share: they act in the universe just *like a **point*** in spacetime.



Instead:



(nothing of its inside is visible)

This picture is in contrast to the idea of Cohl Furey, that elementary particles may be viewed as ideals[2] in the tensor algebra of division algebras over the reals $\mathbb{R}\otimes\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ (see [5]).
As for example, Furey could identify the group structure of the standard model via ideals of $\mathbb{C}\otimes\mathbb{O}$. Now, ideals in such a structure essentially are defined via the multiplication of participating algebras. Why is the multiplication so essential? We know, that the algebras ℂ, ℍ and 𝕆 are the only finite dimensional division algebras over the reals, although their principle of construction, the Cayley-Dickson-doubling process, may continue beyond dimension of 8. The constructed algebras loose identities with higher dimensions. First (ℝ→ℂ) ordering is lost, then (ℂ → ℍ) commutativity of multiplication is lost, next (ℍ → 𝕆) associativity of multiplication is lost, too. Within 𝕆 there remains an identity weaker then associativity, the so-called *alternative identities*: $\boldsymbol{a}(\boldsymbol{ab}) = \boldsymbol{a}^2\boldsymbol{b}\ and\ (\boldsymbol{ab})\boldsymbol{b} = \boldsymbol{ab}^2$ for any two elements $\boldsymbol{a}, \boldsymbol{b}$ of the algebra. But for all algebras over the reals, constructed by the Cayley-Dickson process, one identity remains: all of these algebras are potency associative, i.e.: $\boldsymbol{a^2a} = \boldsymbol{aa^2}$ for any element $\boldsymbol{a}$, and more general satisfy the so-called *flexible identity*: $\boldsymbol{a}(\boldsymbol{ba}) = (\boldsymbol{ab})\boldsymbol{a}$ and as quadratic algebra the Jordan identity $\boldsymbol{a}^2(\boldsymbol{ba}) = (\boldsymbol{a}^2\boldsymbol{b})\boldsymbol{a}$ for any two elements $\boldsymbol{a}, \boldsymbol{b}$ of the algebra (see [9]). So, basics of the ideas to come is, that it is not the multiplication in general, but the potencies of elements, that will lead us to special elements within the above algebras and those being of higher dimension than 8 and

2 an ideal in $\mathbb{C}\otimes\mathbb{O}$ (subalgebra of $\mathbb{R}\otimes\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$) is a subalgebra, that is invariant under multiplication by any element of the total algebra, i.e. multiplication by some element again yields an element of the subalgebra.

having zero divisors. Basically, we will be interested in *roots of unity*, i.e. elements with a potency leading to 1 (neutral element). The key reason is the idea of a lifecycle of such elements, joining our universe at lifecycle points, where the potency is ±1 (part of ordered structure $\mathbb{R}$ (time)). Roughly spoken:

»*An object without a periodical internal lifecycle will not survive*« (the expansion phase of our universe, since it does not interact) [except when it is part of some periodical process (destroyed and created), that is stable]

Suppose, that an element $\boldsymbol{a}$ is such, that $\boldsymbol{a}^{n} = 1$, then a lifecycle defined by $\boldsymbol{a}^{i}\boldsymbol{a}$ (i=1,…,n) will be periodical. For elements of $\mathbb{H}$ we can extend to: $\boldsymbol{a}^{i}\boldsymbol{x}$ (i=1,…,n) will be periodical for an arbitrary element $\boldsymbol{x}\in\mathbb{H}$. For elements of $\mathbb{O}$ we can extend to: $\boldsymbol{a}^{2^{i}}\boldsymbol{x}$ (i=1,…,m and n=2^m) will be periodical for an arbitrary element $\boldsymbol{x}\in\mathbb{O}$. If we go beyond using the Cayley-Dickson-process to construct algebra of dimension 2^n ($n\in\mathbb{N}$) over the reals, like the algebra $\mathbb{S}$ of ***sedenions*** with dimension 16 (see [7]) and further on, periodicity identity is valid only for $\boldsymbol{a}^{i}\boldsymbol{a}$. But note, that, in the algebra of *sedenion*s, there will be 3 special subalgebras. Two of them will be isomorphic to $\mathbb{O}$, the third one will contain zero divisors. All the three of them (as 8-dim. vector spaces) share $\mathbb{H}$ as a subalgebra and cover the complete 16-dimensional vector space of the *sedenion*s. The algebra of *sedenion*s, although it already has zero divisors and therefore is no longer a division algebra, will be the fundamental algebra, when implementing the standard model of elementary particles in the theory on black holes as the only objects within our universe[3]. Due to a famous theorem of Hurwitz we know, that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are the only finite dimensional division algebras over the reals and also the only *normed* algebras, that contain a quadratic form compatible with the algebra's multiplication. We will come back to all of this later.

Suppose, that an element within our universe does not have a periodic lifecycle. Then there are 2 possibilities: it ends or its lifecycle does not end and never "joins the reals". In the first case, the element has vanished in the early universe. In the second case, it is there, but since there are no interactions with the rest, we will never "see" it, which is the same as if it does not exist at all in our universe.

Since we assume, that elements in our universe are black holes (with respect to curvature of spacetime), the black holes (including elementary particles) in our universe are assumed to have kind of an "inner life", in fact a periodical one, that will be given by potencies of $(\boldsymbol{a})^{2^{i}}$ for 0≤i≤n, n>0, where $\boldsymbol{a} \in \mathbb{O}$ with $(\boldsymbol{a})^{2^{n}} = -1 \; and \; (\boldsymbol{a})^{2^{n-1}} \neq \pm 1$. So, $(\boldsymbol{a})^{2^{n+1}} = 1$, and if the lifecycle is given by $(\mathfrak{l})$ $\mathbb{H} \ni \boldsymbol{x} \to \boldsymbol{ax} \to \boldsymbol{a}^{2}\boldsymbol{x} \to \cdots \to \boldsymbol{a}^{2^{n+1}}\boldsymbol{x} = \boldsymbol{x}$ or $(\mathfrak{r})$ $\mathbb{H} \ni \boldsymbol{x} \to \boldsymbol{xa} \to \boldsymbol{xa}^{2} \to \cdots \to \boldsymbol{xa}^{2^{n+1}} = \boldsymbol{x}$ or $(\mathfrak{s})$ $\mathbb{H} \ni \boldsymbol{x} \to \boldsymbol{axa} \to \boldsymbol{a}^{2}\boldsymbol{xa}^{2} \to \cdots \to \boldsymbol{a}^{2^{n}}\boldsymbol{xa}^{2^{n}} = \boldsymbol{x}$ or $(\mathfrak{i})$ $\mathbb{H} \ni \boldsymbol{x} \to \boldsymbol{axa}^{-1} \to \boldsymbol{a}^{2}\boldsymbol{xa}^{-2} \to \cdots \to \boldsymbol{a}^{2^{n}}\boldsymbol{xa}^{-2^{n}} = \boldsymbol{x}$ this will lead to 4 strictly periodical lifecycles of $\boldsymbol{x} \in \mathbb{O}$. Note that brackets in multiplications may be omitted, although multiplication is not associative, since in $\mathbb{O}$ the identities $\boldsymbol{a}(\boldsymbol{ab}) = (\boldsymbol{aa})\boldsymbol{b} \; and \; (\boldsymbol{ab})\boldsymbol{b} = \boldsymbol{a}(\boldsymbol{bb})$ $(A\text{: } alternative \; identities) and$ $\boldsymbol{a}(\boldsymbol{ba}) = (\boldsymbol{ab})\boldsymbol{a}$ *(F: flexible identity)* and (derived from the alternative identities; see [6]) the so-called Moufang identities (M1): $\big(\boldsymbol{a}(\boldsymbol{ba})\big)\boldsymbol{c} = \boldsymbol{a}\big(\boldsymbol{b}(\boldsymbol{ac})\big) \; and \; (M2)\text{: } (\boldsymbol{ab})(\boldsymbol{ca}) = \boldsymbol{a}((\boldsymbol{bc})\boldsymbol{a})$ are valid for any two respectively three elements $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ of $\mathbb{O}$ (see [6]). Therefore:

[3] Remember, that black holes in the context here are black holes with respect to curvature, not just gravity

type $\mathfrak{s}$:

$(\boldsymbol{a}^{k}\boldsymbol{x})\boldsymbol{a}^{k} \overset{(F)}{=} \boldsymbol{a}^{k}(\boldsymbol{x}\boldsymbol{a}^{k})$ for any k∈ℕ. Additionally, we get:

$(\boldsymbol{a}(\boldsymbol{a}\boldsymbol{x}\boldsymbol{a}))\boldsymbol{a} \overset{(F)}{=} \boldsymbol{a}\big((\boldsymbol{a}(\boldsymbol{x}\boldsymbol{a}))\boldsymbol{a}\big) \overset{(F)}{=} \boldsymbol{a}\big(\boldsymbol{a}((\boldsymbol{x}\boldsymbol{a})\boldsymbol{a})\big) \overset{(A)}{=} (\boldsymbol{a}\boldsymbol{a})((\boldsymbol{x}\boldsymbol{a})\boldsymbol{a}) \overset{(A)}{=} \boldsymbol{a}^2(\boldsymbol{x}\boldsymbol{a}^2) \overset{(F)}{=} (\boldsymbol{a}^2\boldsymbol{x})\boldsymbol{a}^2 = \boldsymbol{a}^2\boldsymbol{x}\boldsymbol{a}^2$. So, $\big(\boldsymbol{a}^2(\boldsymbol{a}^2\boldsymbol{x}\boldsymbol{a}^2)\big)\boldsymbol{a}^2 = \boldsymbol{a}^4\boldsymbol{x}\boldsymbol{a}^4$, and so on. So, the process of type $\mathfrak{s}$ can be *derived by multiplication* of $\boldsymbol{a}^{2^k}$ on both sides of $\boldsymbol{a}^{2^k}\boldsymbol{x}\boldsymbol{a}^{2^k}$ to get $\boldsymbol{a}^{2^{k+1}}\boldsymbol{x}\boldsymbol{a}^{2^{k+1}}$. This is not clear for:

type $\mathfrak{i}$:

$(\boldsymbol{a}^{k}\boldsymbol{x})\boldsymbol{a}^{-k} = (\boldsymbol{a}^{-k}\boldsymbol{a}^{k})((\boldsymbol{a}^{k}\boldsymbol{x})\boldsymbol{a}^{-k}) \overset{(M2)}{=} \boldsymbol{a}^{-k}((\boldsymbol{a}^{k}(\boldsymbol{a}^{k}\boldsymbol{x}))\boldsymbol{a}^{-k}) \overset{(A)}{=} \boldsymbol{a}^{-k}((\boldsymbol{a}^{2k}\boldsymbol{x})\boldsymbol{a}^{-k}) \overset{(M2)}{=} (\boldsymbol{a}^{-k}\boldsymbol{a}^{2k})(\boldsymbol{x}\boldsymbol{a}^{-k}) = \boldsymbol{a}^{k}(\boldsymbol{x}\boldsymbol{a}^{-k})$ for any k∈ℕ. Whether $(\boldsymbol{a}^{2^k}((\boldsymbol{a}^{2^k}\boldsymbol{x})\boldsymbol{a}^{-2^k}))\boldsymbol{a}^{-2^k} = \boldsymbol{a}^{2^{k+1}}\boldsymbol{x}\boldsymbol{a}^{-2^{k+1}} = \boldsymbol{a}^{2^k}((\boldsymbol{a}^{2^k}(x\boldsymbol{a}^{-2^k}))\boldsymbol{a}^{-2^k})$ with respect to $(\boldsymbol{a})^{2^n} = -1$ *and* $(\boldsymbol{a})^{2^{n-1}} \neq \pm 1$ is an open question. We will show, that for so-called *roots of unity* this is true.

Other types may be handled in a similar way (using (A)). Additionally, if $\boldsymbol{a}$ satisfies $(\boldsymbol{a})^{2^n} = -1$ *and* $(\boldsymbol{a})^{2^{n-1}} \neq \pm 1$, then $-\boldsymbol{a}$ *and* $\boldsymbol{a}^{-1}$ do so, too.

Note, that the alternative identities and within them the Moufang identities do no longer hold on for algebras derived via the Cayley-Dickson doubling process beyond $\mathbb{O}$ like *sedenion*s etc., but for the 16-dimensional algebra of *sedenion*s $\mathbb{S}$ (F) is still true, meaning, that periodical lifecycles of types ($\mathfrak{l}$), ($\mathfrak{r}$) and ($\mathfrak{s}$) may still exist, but not in a *derivation by multiplication* (see remark on type $\mathfrak{s}$ above). Although $\mathbb{S}$ has zero divisors, 2^n-potencies (n∈ℕ) of an element $\boldsymbol{a} \in \mathbb{S}$ may only be zero, if $\boldsymbol{a} = \boldsymbol{0}$. This is because for $\boldsymbol{a} = \sum_{i=0}^{15} a_i\boldsymbol{e}_i$ $(a_i \in \mathbb{R})$ *with* $(\boldsymbol{e}_i^{\boldsymbol{2}} = -\boldsymbol{e}_0$ *and* $\boldsymbol{e}_i\boldsymbol{e}_j = -\boldsymbol{e}_j\boldsymbol{e}_i$ *for* $i, j \in \{1, \ldots, 15\}, i < j)$ *and* $r^2 = \sum_{i=1}^{15} a_i^2$: $\boldsymbol{a}^{\boldsymbol{2}} = (a_0^2 - r^2)\boldsymbol{e}_0 + 2a_0\sum_{i=1}^{15} a_i\boldsymbol{e}_i$, which may only be zero for all the a_i being zero. Note also, that there will be three 8-dimensional subalgebras (called generations $\mathbb{O}_1, \mathbb{O}_2, \mathbb{O}_3$) of the sedenions $\mathbb{S}$, that share $\mathbb{H}$, and – as vector spaces - together build the complete underlying vector space of the *sedenions* $\mathbb{S}$.

In appendix A there will be proved, that if $\boldsymbol{a} \neq \boldsymbol{0}$ is element of one of the 3 generations and $\boldsymbol{x} \in \mathbb{H}, \boldsymbol{x} \neq \boldsymbol{0}$, then $\boldsymbol{a}\boldsymbol{x} \neq \boldsymbol{0}, \boldsymbol{x}\boldsymbol{a} \neq \boldsymbol{0}, \boldsymbol{a}\boldsymbol{x}\boldsymbol{a} \neq \boldsymbol{0}$ *and* $\boldsymbol{a}\boldsymbol{x}\boldsymbol{a}^* \neq \boldsymbol{0}$, where $\boldsymbol{a}^*$ is the conjugate of $\boldsymbol{a}$ ($\boldsymbol{a}\boldsymbol{a}^* \in \mathbb{R}$ is the Euclidian norm of $\boldsymbol{a}$ and $\boldsymbol{a}^{-\boldsymbol{1}} = \frac{1}{\boldsymbol{a}\boldsymbol{a}^*}\boldsymbol{a}^*$ for $\boldsymbol{a} \neq \boldsymbol{0}$).

Note, that the “exponential” steps in lifecycles need not to be the only ones. In fact, a process of say type ($\mathfrak{l}$) $\mathbb{H} \ni \boldsymbol{x} \to \boldsymbol{a}\boldsymbol{x} \to \boldsymbol{a}^2\boldsymbol{x} \to \cdots \to \boldsymbol{a}^{2^{n+1}}\boldsymbol{x} = \boldsymbol{x}$ may be accomplished by repeatedly multiplying $\boldsymbol{a}$‘s from the left ($\mathfrak{l}$) $\mathbb{H} \ni \boldsymbol{x} \to \boldsymbol{a}\boldsymbol{x} \to \boldsymbol{a}(\boldsymbol{a}\boldsymbol{x}) \to \cdots \to \boldsymbol{a}(\boldsymbol{a}(\ldots.(\boldsymbol{a}(\boldsymbol{a}\boldsymbol{x}))\ldots))$ and then recursively applying rules (A) to get $\to \boldsymbol{a}^{2^n}\boldsymbol{x} = -\boldsymbol{x}$, and then again $\mathbb{H} \ni -\boldsymbol{x} \to -\boldsymbol{a}\boldsymbol{x} \to -\boldsymbol{a}(\boldsymbol{a}\boldsymbol{x}) \to \cdots \to -\boldsymbol{a}(\boldsymbol{a}(\ldots.(\boldsymbol{a}(\boldsymbol{a}\boldsymbol{x}))\ldots))$ to get $\to -\boldsymbol{a}^{2^n}\boldsymbol{x} = \boldsymbol{x}$, completely: $\boldsymbol{a}^{2^{n+1}}\boldsymbol{x} = \boldsymbol{x}$. So, the lifecycle shall not be interpreted as being (exponentially) accelerated.

ℝ (ordered) shall be viewed as time axis. So, objects of type ($\mathfrak{l}$) and ($\mathfrak{r}$) with $\boldsymbol{a} \in \mathbb{O}$ will join ongoing time in our universe at 2 points, the first, when reaching $(\boldsymbol{a})^{2^n} = -1$, the second one when reaching $(\boldsymbol{a})^{2^{n+1}} = 1$. This is somehow strange, since $(\boldsymbol{a})^{2^n} = -1$ will stand for a point in the past, whereas $(\boldsymbol{a})^{2^{n+1}} = 1$ will symbolize the next step (future) on our time axis. For objects of type ($\mathfrak{s}$) and ($\mathfrak{i}$) there is only one point, 1, where they join the reals.

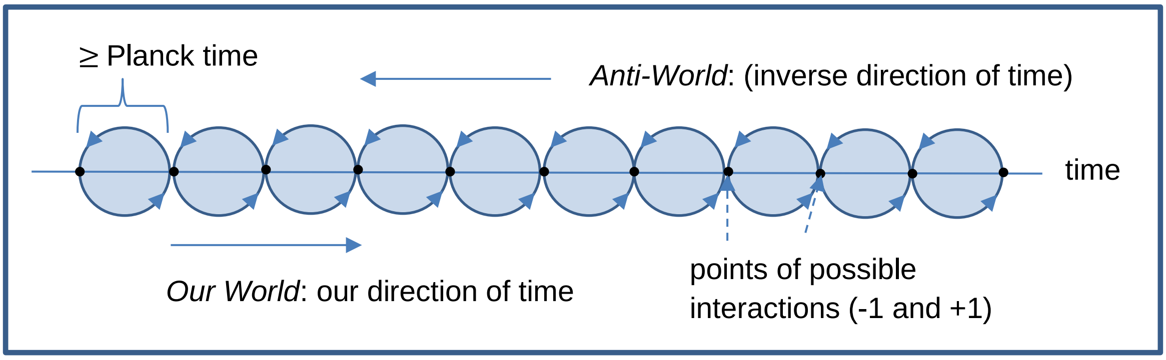
SPIN: Objects with lifecycles of types $(\mathfrak{l})$ and $(\mathfrak{r})$ are said to have spin $\frac{1}{2}$, since they join the reals half a round on the 7-sphere at -1 and next at 1. Objects with lifecycles of types $(\mathfrak{s})$ and $(\mathfrak{i})$ in this design have spin 1. Other spins do not exist. One may define spin alternatively as turnarounds (the clock) before joining the reals at -1 and/or 1. This is not done here.

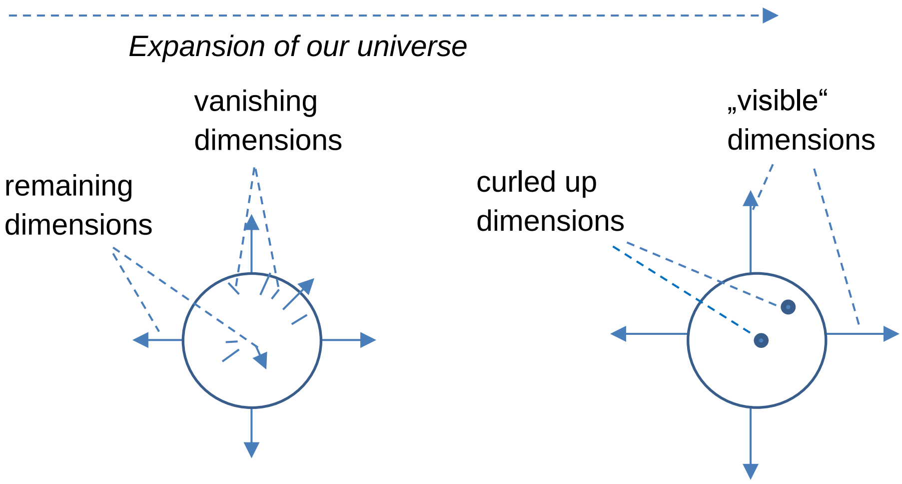
Remember, that for transformations of kind ◎* on elements of $\mathbb{H}$: $r' = \gamma l(r - \frac{v}{c}ct)$, $ct' = \gamma l(ct - \frac{v}{c}r)$, θ' = θ, φ' = φ, with *v in direction of r not constant,* we get:

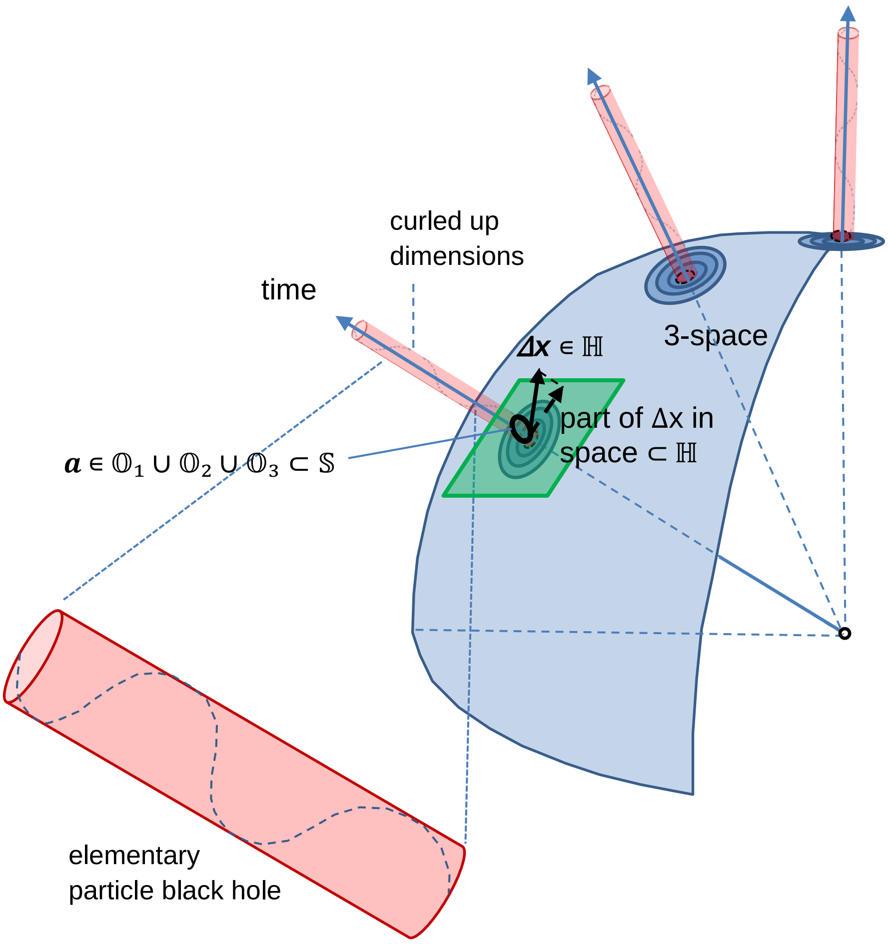
$$\left(\partial r' - \gamma'^2 \frac{\partial v'}{c} ct'\right)^2 - \left(\partial(ct') - \gamma'^2 \frac{\partial v'}{c} r'\right)^2 + \gamma'^2(r' - \frac{v'}{c}ct')^2\left(\partial\theta'^2 + sin^2\theta'\partial\varphi'^2\right) =$$
$$l^2[\left(\partial r - \gamma^2 \frac{\partial v}{c} ct\right)^2 - \left(\partial(ct) - \gamma^2 \frac{\partial v}{c} r\right)^2 + \gamma^2(r - \frac{v}{c}ct)^2\left(\partial\theta^2 + sin^2\theta\partial\varphi^2\right)]$$

The special case $l^2 = 1$ may be accomplished in real numbers by $l = \pm 1$. As already mentioned, the idea is, that in the beginning of our universe at big bang higher dimensions have been created, and within associated algebras the identity $l^{2^n} = 1$ (applying transformation multiple times) has been fulfilled in several ways: $l = \boldsymbol{a}^{2^n}$, n>0, where $\boldsymbol{a} \in \mathbb{O}$ with $\boldsymbol{a}^{2^n} = -1$ *and* $\boldsymbol{a}^{2^{n-1}} \neq \pm 1$. Potency associative Algebras beyond the *sedenion*s with dimensions higher than 16 "vanished" because of including zero divisors and not fulfilling derivation by multiplication for periodical lifecycles of types $(\mathfrak{l})$, $(\mathfrak{r})$, $(\mathfrak{s})$ and $(\mathfrak{i})$, and only the known finite dimensional division algebras over the real numbers: $\mathbb{R}$, $\mathbb{C}$ (complex numbers), $\mathbb{H}$ (quaternions) and $\mathbb{O}$ (octonions) remained (as subalgebras of the »container-algebra« $\mathbb{S}$ of sedenions) and their dimensions higher than 4 were curled up – within special black holes (with respect to curvature, not just gravity). This also would mean, that parts of the complete transformation $(l^2, l^4, ..)$ will be hidden. An object (black hole), curling up such higher dimensions will "join" our universe only after some internal process, just as assumed for black holes of type universe, but with a different internal periodic process (see [2]). One could argue, that the 7-dimensional spacelike imaginary part of the octonions $\mathbb{O}$ (generation I) is separated from the space of our universe and only 1-dimensional real time is binding both. In this picture, our universe together with the elementary particle black holes of generation I would be part of an 11-dimensional space. In the model presented here, the imaginary part of the elementary particle black holes and the space of our universe shall not be separated. So, the total dimension is 8 in this model with space part of dimension 3+4=7. 3 for the space of our universe and 4 for curled up dimensions of special elementary particles of generation I (similar in the other two generations represented by the 2 subalgebras of the sedenions, one isomorphic to $\mathbb{O}$, the other one with zero divisors, where the zero divisors are not relevant to the constructed elements based on so-called *roots of unity* and their interactions).

Note, that in this picture we are just part of assemblies of elementary particles, that interact with each other. And when they do, time will go on one step forward (for the particles interacting, not the universe as a whole). That is, our view will change. Or, as we are shapes of views, one of the associated views is changing. Change may happen according to (mathematically) continuous identities, valid only at interaction points (see [12]). In the picture to come an Anti-World is mentioned. Note, that this is an additive inverse of our world. So, if particles of both worlds would happen to meet, nothing would remain, not even radiation. The anti-world would not be made from what we call anti-particles. These are part of our world. So, the first symmetry rule, that our universe obeys, is: Do not mix particles of both worlds. More on this concept will be found in appendix B. The following two pictures show the ongoing process and space-embedding:


≥ Planck time
Anti-World: (inverse direction of time)
time
Our World: our direction of time
points of possible
interactions (-1 and +1)


Expansion of our universe
vanishing
dimensions
remaining
dimensions
curled up
dimensions
„visible“
dimensions


curled up
dimensions
time
$\Delta\boldsymbol{x} \in \mathbb{H}$
3-space
part of Δx in
space $\subset \mathbb{H}$
$\boldsymbol{a} \in \mathbb{O}_1 \cup \mathbb{O}_2 \cup \mathbb{O}_3 \subset \mathbb{S}$
elementary
particle black hole

If $\boldsymbol{a}$ would be an element of a higher dimensional algebra like the *sedenions*, it could be such that $\boldsymbol{a}$ *and* $\boldsymbol{x}$ would be part of a subalgebra isomorphic to $\mathbb{O}$. But the world line of $\boldsymbol{x}$ (within spacetime $\mathbb{H}$) would change according to transformations of G15*. So, the spanned subalgebra would not necessarily stay to be isomorphic to $\mathbb{O}$. But it would, if the subalgebra is one of the two corresponding to generations I-II, as we will see. The argument obviously depends on our pre-condition, that spacetime and the dimensions of the black hole of type elementary particle shall not be decoupled. Generally, loss of periodicity leads to instability in time and loss of interaction.

Usually, additional dimensions enlarge a space at each of its points. This is not necessarily true for a discrete space (=spacetime), which our universe here is assumed to be, since then continuous mathematics is only an approximation of reality. So, additional dimensions of our 4-dimensional spacetime may only occur within its black holes.

Now, let us have a look on 2^n-potencies of some element $\boldsymbol{a} = \sum_{i=1}^{7} a_i \boldsymbol{e}_i$ of $\mathbb{O}$, $a_i \in \mathbb{R}, 0 \le i \le 7$, with the $\boldsymbol{e}_i$ building a basis of the 7-dimensional imaginary part, satisfying the multiplication of algebra $\mathbb{O}$ according to table taken from [5].

		e_j							
	$e_i e_j$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_i	e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
	e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
	e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
	e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
	e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
	e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
	e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
	e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

$\boldsymbol{e}_0$ is taken to yield time (coordinate: ct), $\boldsymbol{e}_1$ to $\boldsymbol{e}_7$ providing the spacelike coordinates of the imaginary part of $\mathbb{O}$ with $r^2 = \sum_{i=1}^{7} {a_i}^2$, $\mathbf{I}(\boldsymbol{a})$ being an abbreviation for $\sum_{i=1}^{7} a_i \boldsymbol{e}_i$. Let $a_0 := ct$.

We get for $\boldsymbol{a} \in \mathbb{O}$ with $r^2 \neq 0$:

$\boldsymbol{a}^2 = a_i a_j \boldsymbol{e}_i \boldsymbol{e}_j = ((ct)^2 - r^2)\boldsymbol{e}_0 + 2ct\,\mathbf{I}(\mathbf{a}) = r^2[(\frac{(ct)^2}{r^2} - 1)\boldsymbol{e}_0 + 2\frac{ct}{r}\mathbf{I}(\frac{\boldsymbol{a}}{r})]$, (since $\boldsymbol{e}_i \boldsymbol{e}_j + \boldsymbol{e}_j \boldsymbol{e}_i = \mathbf{0}$ for 1≤i<j≤7). So, $\left(\frac{a}{r}\right)^2 = (\frac{(ct)^2}{r^2} - 1)\boldsymbol{e}_0 + 2\frac{ct}{r}\mathbf{I}(\frac{\boldsymbol{a}}{r})$.

Note, that all the potencies of $\boldsymbol{a}$ are located in a plane, spanned by $\{\ \boldsymbol{e}_0\ ,\ \mathbf{I}(\boldsymbol{a})\}$.

For $\boldsymbol{a} := \sum_{i=0}^{7} a_i \boldsymbol{e}_i \in \mathbb{O}$, $\boldsymbol{a}^* := a_0 \boldsymbol{e}_0 - \sum_{i=1}^{7} a_i \boldsymbol{e}_i$ one defines: $|\ \boldsymbol{a}\ |^2 := \boldsymbol{a}\boldsymbol{a}^* = (\sum_{i=0}^{7} {a_i}^2)\boldsymbol{e}_0$ and gets a quadratic form, that is compatible with the multiplication: $|\boldsymbol{ab}|=|\boldsymbol{a}||\boldsymbol{b}|$. Here, usually $\boldsymbol{e}_0$ is identified by $\mathbf{1} \in \mathbb{R}$.

Notation:

If we write $\mathbb{C}$ as a subalgebra of $\mathbb{O}$ without naming its generating base, we mean the subalgebra generated by $\boldsymbol{e}_1$ according to the multiplication table above. If we write $\mathbb{H}$ as a subalgebra of $\mathbb{O}$ without naming its generating base, we mean the subalgebra generated by $\boldsymbol{e}_1, \boldsymbol{e}_2$ according to the multiplication table above.

From [4] we take the facts:

Any subalgebra of $\mathbb{O}$ generated by $\boldsymbol{e}_i$ (i≠0) is isomorphic to $\mathbb{C}$. Any subalgebra of $\mathbb{O}$ generated by $\boldsymbol{e}_i, \boldsymbol{e}_j$ (i,j≠0, i≠j) is isomorphic to $\mathbb{H}$. And any subalgebra of $\mathbb{O}$ generated by $\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k$ (i,j,k≠0, i<j<k, and none of the triple elements is part of the subalgebra generated by the other two elements) is $\mathbb{O}$ itself.

From [3] we take the following remarkable results:

Hurwitz: ℝ, ℂ, ℍ and 𝕆 are the only finite dimensional division algebras over the reals. The next candidate in the so-called Cayley-Dickson construction by defining a multiplication of the form $(\boldsymbol{a},\boldsymbol{b})*(\boldsymbol{c},\boldsymbol{d}) := (\boldsymbol{ac}-(\boldsymbol{d}^*)\boldsymbol{b}, \boldsymbol{da}+\boldsymbol{b}(\boldsymbol{c}^*))$ on pairs of elements of division algebra 𝕆 over the reals, the *sedenions* 𝕊, already leads to an algebra having zero divisors (see [7]).

Roots of unity

Let us inspect elements $\boldsymbol{a}\in\mathbb{O}$ with the norm of its imaginary part $r^2 = |\boldsymbol{I}(\boldsymbol{a})|^2 \neq 0$, such that $\boldsymbol{a}^{2^n} = -1*\boldsymbol{e}_0$ *and* $\boldsymbol{a}^{2^{n-1}} \neq \pm 1*\boldsymbol{e}_0$ for some natural number n>1. Then we call $\boldsymbol{a}$ a *root of unity (of degree n of generation I)*.

Note, that for a *root of unity (of degree n of generation I)* we have $\boldsymbol{a}^{2^{n-1}} \in \mathbb{H}$ if and only if $\boldsymbol{a} \in \mathbb{H}$. Let $\boldsymbol{b} := \boldsymbol{a}^{2^{n-2}} := \sum_{i=0}^{7} b_i\boldsymbol{e}_i$, with $b_i \in \mathbb{R}$ $(0 \le i \le 7)$, then $\boldsymbol{b}^2 = ({b_0}^2 - |\boldsymbol{I}(\boldsymbol{b})|^2)\boldsymbol{e}_0 + 2b_0\sum_{i=1}^{7} b_i\boldsymbol{e}_i \in \mathbb{H} \Rightarrow b_i = 0$ for $4 \le i \le 7$, since $b_0 \neq 0$ (otherwise: $\boldsymbol{b}^2 = -|\boldsymbol{I}(\boldsymbol{b})|^2\boldsymbol{e}_0$, but $\boldsymbol{b}^4 = \boldsymbol{a}^{2^n} = -1*\boldsymbol{e}_0$, a contradiction).

Note, that $r^2 \neq 1$, since $\boldsymbol{a}^2 = ((ct)^2 - r^2)\boldsymbol{e}_0 + 2ct\,\mathbf{I}(\boldsymbol{a})$, and $|\boldsymbol{a}^{2^{n+1}}| = |\boldsymbol{a}^2|^{2^n} = 1$. So, $|\boldsymbol{a}^2|^2 = \boldsymbol{a}^2(\boldsymbol{a}^2)^* = ((ct)^2 - r^2)^2 + 4ct^2r^2 = ((ct)^2 + r^2)^2$ *and* $\mathbb{R}_+ \ni |\boldsymbol{a}^2| = 1$, which yields $(ct)^2 + r^2 = 1$. If $r^2 = 1$, we would get ct=0 and therefore $\boldsymbol{a}^2 = -1*\boldsymbol{e}_0$, which contradicts n>1. So, $(ct)^2 + r^2$ must be of the form $cos^2\varphi + sin^2\varphi$ for some φ with 0<φ<π.

We find, that a root of unity $\boldsymbol{a}\in\mathbb{O}$ may be written as $\boldsymbol{a} = cos\varphi\boldsymbol{e}_0 \pm sin\varphi\sum_{i=1}^{7} a_i\boldsymbol{e}_i$ with $\left|\sum_{i=1}^{7} a_i\boldsymbol{e}_i\right|^2 = \sum_{i=1}^{7} {a_i}^2 = 1$ and $\varphi = (2m-1)\frac{\pi}{2^n}$, m∈{0,…,$2^{n-2}$} for some n∈ℕ, n>1.[4]

Definition »*summed root of unity (of generation I)*«:

Let {$\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{e}_6, \boldsymbol{e}_7$} be a base for the 8-dimensional vector space over the reals with multiplication according to the above multiplication table of octonions 𝕆. Let $z_i\in\mathbb{Z}$ (i=1,..,7), ℕ∋n of them not being 0. Then the weighted sum $\boldsymbol{z} := \frac{1}{n}\sum_{i=1}^{7} z_i\boldsymbol{e}_i$ shall be called a *summed root of unity of generation I*, if $\frac{1}{n}\sum_{i=1}^{7} z_i \in \{0, \pm 1\}$ and $\frac{1}{n^2}\sum_{i=1}^{7} {z_i}^2 = 1$.

In case that $\sum_{i=1}^{7} z_i = 0$ we allow $n \in \{\sqrt{2}, \sqrt{6}\}$. [5]

Note, that if $\boldsymbol{z}$ is a summed root of unity, then $\boldsymbol{z}^2 = -\boldsymbol{e}_0$.

Examples (with respect to base B_1:={$\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{e}_6, \boldsymbol{e}_7$} of $\boldsymbol{I}(\mathbb{O}_1)$ above:

n=1: Take $\boldsymbol{e} \in B$ and set $\boldsymbol{z} := \pm\boldsymbol{e}$

n=2: Let $\{\boldsymbol{e}_i, \boldsymbol{e}_j\} \subset B$, i≠j, and set $\boldsymbol{z} := \frac{1}{\sqrt{2}}(\boldsymbol{e}_i - \boldsymbol{e}_j)$

n=3: Let $\{\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k\} \subset B$, i,j,k distinct, and set $\boldsymbol{z} := \frac{1}{3}(2\boldsymbol{e}_i + 2\boldsymbol{e}_j - \boldsymbol{e}_k)$;

Or set $\boldsymbol{z} := \frac{1}{\sqrt{6}}(2\boldsymbol{e}_i - \boldsymbol{e}_j - \boldsymbol{e}_k)$

One easily verifies, that there are no other solutions for these values of n.

4 The condition on φ ensures that a stepwise internal lifecycle meets the reals so that an interaction with spacetime may take place

5 The reason is, that elements may act only pairwise or together with its conjugate (spin=1) on space or later defined elementary particles, and then $\frac{1}{n}*\frac{1}{n} = \frac{1}{n^2}$ with $n^2 \in \mathbb{N}$, and n=2, n=6 are the only even numbers within 1,..,7, that yield a square root not part of ℕ.

Velocity and roots of unity

Let $\boldsymbol{\partial x} = \sum_{i=0}^{3} \partial x_i \boldsymbol{e}_i \in \mathbb{H}$, $\partial x_i \in \mathbb{R}$, $\boldsymbol{e}_i$ building a basis of $\mathbb{H}$, $0 \le i \le 3$, and $\partial(ct) := \partial x_0$, using the multiplication in $\mathbb{H}$. We got:

$$\boldsymbol{\partial x^2} = (\partial(ct))^2 - ({\partial x_1}^2 + {\partial x_2}^2 + {\partial x_3}^2))\boldsymbol{e}_0 + 2\partial(ct)(\partial x_1 \boldsymbol{e}_1 + \partial x_2 \boldsymbol{e}_2 + \partial x_3 \boldsymbol{e}_3)$$

The time coordinate ($\boldsymbol{e}_0$) just describes the ordinary Minkowski metric, the space part kind of an expansion of space. We can derive:

$\left(\frac{\boldsymbol{\partial x}}{\partial t}\right)^2 = (c^2 - v^2)\boldsymbol{e}_0 + 2c\boldsymbol{v}$ with $\boldsymbol{v} := \frac{\partial x_1}{\partial t}\boldsymbol{e}_1 + \frac{\partial x_2}{\partial t}\boldsymbol{e}_2 + \frac{\partial x_3}{\partial t}\boldsymbol{e}_3$ and $v := |\boldsymbol{v}|$. So:

$$\frac{1}{c^2}\left(\frac{\boldsymbol{\partial x}}{\partial t}\right)^2 = (1 - \frac{v^2}{c^2})(\boldsymbol{e}_0 + \frac{2\frac{v}{c}}{1-\frac{v^2}{c^2}} * \frac{\boldsymbol{v}}{v})$$

The value of $\frac{2\frac{v}{c}}{1-\frac{v^2}{c^2}}$ increases monotonal with $v \in [0, c[$, so we may set: $tan2\varphi := \frac{2\frac{v}{c}}{1-\frac{v^2}{c^2}}$ or $2\varphi = \arctan\frac{2\frac{v}{c}}{1-\frac{v^2}{c^2}}$, with $0 \le \varphi < \frac{\pi}{4}$. Using $\cos(\arctan(r)) = \pm\frac{1}{\sqrt{1+r^2}}$ and $\gamma^2 := \frac{1}{1-\frac{v^2}{c^2}}$, we get:

$$\frac{1}{c^2}\left(\frac{\boldsymbol{\partial x}}{\partial t}\right)^2 = \frac{1-\frac{v^2}{c^2}}{\cos\left(\arctan\frac{2\frac{v}{c}}{1-\frac{v^2}{c^2}}\right)}(cos2\varphi\boldsymbol{e}_0 + sin2\varphi * \frac{\boldsymbol{v}}{v}) = \frac{\frac{1}{\gamma^2}}{\cos\left(\arctan(2\gamma^2\frac{v}{c})\right)}(cos2\varphi\boldsymbol{e}_0 + sin2\varphi * \frac{\boldsymbol{v}}{v}) =$$

$$\frac{\sqrt{1+4\gamma^4\frac{v^2}{c^2}}}{\gamma^2}(cos2\varphi\boldsymbol{e}_0 + sin2\varphi * \frac{\boldsymbol{v}}{v})$$

Finally we get:

$\frac{1}{c^2}\left(\frac{\boldsymbol{\partial x}}{\partial t}\right)^2 = (1 + \frac{v^2}{c^2})(cos2\varphi\boldsymbol{e}_0 + sin2\varphi * \frac{\boldsymbol{v}}{v}) = (1 + \frac{v^2}{c^2})(cos\varphi\boldsymbol{e}_0 + sin\varphi * \frac{\boldsymbol{v}}{v})^2$ with coupling: $\varphi = \frac{1}{2}\arctan(2\gamma^2\frac{v}{c})$.

This way there is kind of an analogy for the change of spacetime and the root of unity concept.

… work to be done …

Generations and colours

In Quantum Theory elementary particles of type fermions are ordered in 3 generations. In order to find these generations, we will have to change to the *sedenions* $\mathbb{S}$. We use the following multiplication table, taken from [7] but enriched by »colours«:

*	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6	e_9	$-e_8$	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$	e_{10}	e_{11}	$-e_8$	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$	e_{11}	$-e_{10}$	e_9	$-e_8$	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3	e_{12}	e_{13}	e_{14}	e_{15}	$-e_8$	$-e_9$	$-e_{10}$	$-e_{11}$
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	$-e_8$	e_{11}	$-e_{10}$
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	$-e_8$	e_9
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	$-e_8$
e_8	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$	$-e_0$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_9	e_9	e_8	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$	$-e_1$	$-e_0$	$-e_3$	e_2	$-e_5$	e_4	e_7	$-e_6$
e_{10}	e_{10}	e_{11}	e_8	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}	$-e_2$	e_3	$-e_0$	$-e_1$	$-e_6$	$-e_7$	e_4	e_5
e_{11}	e_{11}	$-e_{10}$	e_9	e_8	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}	$-e_3$	$-e_2$	e_1	$-e_0$	$-e_7$	e_6	$-e_5$	e_4
e_{12}	e_{12}	e_{13}	e_{14}	e_{15}	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_4$	e_5	e_6	e_7	$-e_0$	$-e_1$	$-e_2$	$-e_3$
e_{13}	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	e_8	e_{11}	$-e_{10}$	$-e_5$	$-e_4$	e_7	$-e_6$	e_1	$-e_0$	e_3	$-e_2$
e_{14}	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	e_8	e_9	$-e_6$	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	$-e_0$	e_1
e_{15}	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	e_8	$-e_7$	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	$-e_0$

The subalgebra generated by $\{e_0, e_1, e_2, e_3\}$ is $\mathbb{H}$. Let us denote the one generated by $\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ as $\mathbb{O}_1$, the one generated by $\{e_0, e_1, e_2, e_3, e_8, e_9, e_{10}, e_{11}\}$ by $\mathbb{O}_2$ and the one generated by $\{e_0, e_1, e_2, e_3, e_{12}, e_{13}, e_{14}, e_{15}\}$ by $\mathbb{O}_3$.

Notation: Let us denote $\mathbb{H}$ as spacetime, the vector spaces generated by $\{e_4, e_5, e_6, e_7\}$ as anti-spacetime I ($\mathbb{G}_1$), the one generated by $\{e_8, e_9, e_{10}, e_{11}\}$ as anti-spacetime II ($\mathbb{G}_2$) and the one generated by $\{e_{12}, e_{13}, e_{14}, e_{15}\}$ as anti-spacetime III ($\mathbb{G}_3$).

We know, that the subalgebras $\mathbb{O}_1$ and $\mathbb{O}_2$ are isomorphic to $\mathbb{O}$. For $\mathbb{O}_3$ this is not true. In fact, it contains zero divisors. Nevertheless, they share $\mathbb{H}$ and any product of elements of $\mathbb{H}\backslash\{0\}$ and $\mathbb{O}_3\backslash\{0\}$ is nonzero. Moreover no potency of an element in $\mathbb{O}_3\backslash\{0\}$ becomes zero. But there is a severe deficit for $\mathbb{O}_3$: It does not satisfy the alternative identities. As for example:

$$(e_1+e_{13})*((e_1+e_{13})*(e_2+e_{14}))=-4(e_2+e_{14})\neq-2(e_2+e_{14})=(e_1+e_{13})^{2}*(e_2+e_{14})$$

The strange thing is, that if and only if in $\boldsymbol{a}(\boldsymbol{a}\boldsymbol{x})$, with $\boldsymbol{a} \in \mathbb{O}_3$ and $\boldsymbol{x} \in \mathbb{H}$, we take $\boldsymbol{a}$ to be of the form $\boldsymbol{a} = \alpha e_0 + \beta \boldsymbol{z}$ with $\alpha, \beta \in \mathbb{R}$ and $\boldsymbol{z}$ either $\boldsymbol{z} = \sum_{i=1}^{3} z_i e_i$ or $\boldsymbol{z} = \sum_{i=12}^{15} z_i e_i$ then we get:

$\boldsymbol{a}(\boldsymbol{ax}) = (\boldsymbol{aa})\boldsymbol{x}$. The same is true for $\boldsymbol{x}(\boldsymbol{aa}) = (\boldsymbol{xa})\boldsymbol{a}$. This is shown in detail in appendix A1. If $\boldsymbol{a}$ is element of the 4-dimensional subalgebra of $\mathbb{S}$ generated by $\{\boldsymbol{e}_0, \boldsymbol{e}_4, \boldsymbol{e}_8, \boldsymbol{e}_{12}\}$ (isomorphic to $\mathbb{H}$), then $\boldsymbol{a}(\boldsymbol{ay}) = (\boldsymbol{aa})\boldsymbol{y}$ for every $\boldsymbol{y} \in \mathbb{S}$. This is shown in appendix A2.

As a consequence, spin-½-objects in generation III may not contain a mixture of colour- and anticolour-elements. This is *not* forbidden for spin-1-objects. If $\boldsymbol{z} \coloneqq \sum_{i=1}^{3} z_i \boldsymbol{e}_i$ and $\boldsymbol{z}^- \coloneqq \sum_{i=1}^{3} z_i \boldsymbol{e}_{12+i}$ then for $\boldsymbol{a} = \boldsymbol{\alpha e}_0 + \beta \boldsymbol{z}$ and $\boldsymbol{a}^- = \boldsymbol{\alpha e}_0 \pm \beta \boldsymbol{z}^-$ we get the identity:

$\boldsymbol{a}(\boldsymbol{x}\boldsymbol{a}^-) = (\boldsymbol{ax})\boldsymbol{a}^-$ (see appendix A1). Since $\boldsymbol{a}^2, (\boldsymbol{a}^-)^2 \in \mathbb{H}$ the identity remains true for higher potencies $(\boldsymbol{a})^{2^n}, (\boldsymbol{a}^-)^{2^n}$. This identity (adapted in a suggestive manner: $\boldsymbol{z}^- = \sum_{i=1}^{3} z_i \boldsymbol{e}_{4+i}$ for $\mathbb{O}_1$ and $\boldsymbol{z}^- = \sum_{i=1}^{3} z_i \boldsymbol{e}_{8+i}$ for $\mathbb{O}_2$) is *not* true for $\mathbb{O}_1$ and $\mathbb{O}_2$. Here we have to use $\boldsymbol{a}$ or $\boldsymbol{a}^*$ instead of $\boldsymbol{a}^-$. Note, that if $\boldsymbol{a} = cos\varphi \boldsymbol{e}_0 + sin\varphi \boldsymbol{z}$ with imaginary part $\boldsymbol{z}$ in $\mathbb{O}_1$ and $\mathbb{O}_2$ and norm of $\boldsymbol{z}$ equal to 1, then $\boldsymbol{a}^{-1} = cos\varphi \boldsymbol{e}_0 - sin\varphi \boldsymbol{z} = \boldsymbol{a}^*$. And therefore, an identity $(\boldsymbol{a}((\boldsymbol{ax})\boldsymbol{a}^{-1}))\boldsymbol{a}^{-1} = \boldsymbol{a}^2\boldsymbol{x}\boldsymbol{a}^{-2} = \boldsymbol{a}((\boldsymbol{a}(x\boldsymbol{a}^{-1}))\boldsymbol{a}^{-1})$ is true, if and only if it is true for $\boldsymbol{a}$ instead of $\boldsymbol{a}^{-1}$ on the right side of $\boldsymbol{x}$, i.e. $(\boldsymbol{a}((\boldsymbol{ax})\boldsymbol{a}))\boldsymbol{a} = \boldsymbol{a}^2\boldsymbol{x}\boldsymbol{a}^2 = \boldsymbol{a}((\boldsymbol{a}(x\boldsymbol{a}))\boldsymbol{a})$.

Note, that: $\boldsymbol{a}(\boldsymbol{x}\boldsymbol{a}^*) = (\boldsymbol{ax})\boldsymbol{a}^*$ for any Cayley-Dickson algebra $\boldsymbol{A}$, with $\boldsymbol{a}, \boldsymbol{x} \in \boldsymbol{A}$. We have $\boldsymbol{a} = a_0\boldsymbol{e}_0 + \boldsymbol{z}$ with $a_0 \in \mathbb{R}$ and $\boldsymbol{z} \in \boldsymbol{A} \ominus \mathbb{R}\boldsymbol{e}_0$ such that $\boldsymbol{a}^* = a_0\boldsymbol{e}_0 - \boldsymbol{z}$. We get:

$\boldsymbol{a}(\boldsymbol{x}\boldsymbol{a}^*) = (a_0\boldsymbol{e}_0 + \boldsymbol{z})(\boldsymbol{x}(a_0\boldsymbol{e}_0 - \boldsymbol{z})) = (a_0\boldsymbol{e}_0 + \boldsymbol{z})(a_0\boldsymbol{x} - \boldsymbol{xz}) = {a_0}^2\boldsymbol{x} + a_0(\boldsymbol{zx} - \boldsymbol{xz}) - \boldsymbol{z}(\boldsymbol{xz}) = {a_0}^2\boldsymbol{x} + a_0(\boldsymbol{zx} - \boldsymbol{xz}) - (\boldsymbol{zx})\boldsymbol{z}$ due to flexible identity. On the other hand:

$(\boldsymbol{ax})\boldsymbol{a}^* = ((a_0\boldsymbol{e}_0 + \boldsymbol{z})\boldsymbol{x})(a_0\boldsymbol{e}_0 - \boldsymbol{z}) = (a_0\boldsymbol{x} + \boldsymbol{zx})(a_0\boldsymbol{e}_0 - \boldsymbol{z}) = {a_0}^2\boldsymbol{x} + a_0(\boldsymbol{zx} - \boldsymbol{xz}) - (\boldsymbol{zx})\boldsymbol{z}$.

That means, we can expand the definitions of »*summed root of unity*« and »*root of unity*« to generations II and III, represented by $\mathbb{O}_2$ and $\mathbb{O}_3$, in a suggestive manner. All of them carry roots of unity, that satisfy periodical lifecycles of types $(\mathfrak{l})$, $(\mathfrak{r})$ and $(\mathfrak{s})$. Note however restrictions on $\mathbb{O}_3$ and that $\mathbb{O}_3$ contains zero divisors (see [7]).

Now, let $\mathbf{R}$ (*RED*) denote the vector space over the reals generated by $\{\boldsymbol{e}_1\}$ and $\mathbf{R}^-$ (*ANTIRED*) the one generated by $\{\boldsymbol{e}_5, \boldsymbol{e}_9, \boldsymbol{e}_{13}\}$, $\mathbf{B}$ (*BLUE*) the one generated by $\{\boldsymbol{e}_2\}$ and $\mathbf{B}^-$ (*ANTIBLUE*) the one generated by $\{\boldsymbol{e}_6, \boldsymbol{e}_{10}, \boldsymbol{e}_{14}\}$ and $\mathbf{G}$ (*GREEN*) the one generated by $\{\boldsymbol{e}_3\}$ and $\mathbf{G}^-$ (*ANTIGREEN*) the one generated by $\{\boldsymbol{e}_7, \boldsymbol{e}_{11}, \boldsymbol{e}_{15}\}$. Let H alias *WHITE* denote the subalgebra generated by $\{\boldsymbol{e}_0, \boldsymbol{e}_4, \boldsymbol{e}_8, \boldsymbol{e}_{12}\}$ then one may easily prove, that H is isomorphic to the subalgebra of quaternions $\mathbb{H}$.

With the multiplication '*' of the *sedenions* $\mathbb{S}$ we get the following relations:

$\mathbf{R}^2$, $(\mathbf{R}^-)^2$, $\mathbf{R} * \mathbf{R}^-$, $\mathbf{R}^- * \mathbf{R}$, $\mathbf{B}^2$, $(\mathbf{B}^-)^2$, $\mathbf{B} * \mathbf{B}^-$, $\mathbf{B}^- * \mathbf{B}$, $\mathbf{G}^2$, $(\mathbf{G}^-)^2$, $\mathbf{G} * \mathbf{G}^-$, $\mathbf{G}^- * \mathbf{G} \subset \mathrm{H}$, where $\mathbf{R}^2$ means all products of any two elements of $\mathbf{R}$, other products similar.

$\mathrm{H} * \mathbf{R}$, $\mathbf{R} * \mathrm{H}$, $\mathrm{H} * \mathbf{R}^-$, $\mathbf{R}^- * \mathrm{H} \subset \mathbf{R} \oplus \mathbf{R}^-$ and
$\mathrm{H} * \mathbf{B}$, $\mathbf{B} * \mathrm{H}$, $\mathrm{H} * \mathbf{B}^-$, $\mathbf{B}^- * \mathrm{H} \subset \mathbf{B} \oplus \mathbf{B}^-$ and
$\mathrm{H} * \mathbf{G}$, $\mathbf{G} * \mathrm{H}$, $\mathrm{H} * \mathbf{G}^-$, $\mathbf{G}^- * \mathrm{H} \subset \mathbf{G} \oplus \mathbf{G}^-$ and

$\mathbf{R} * \mathbf{B}$, $\mathbf{B} * \mathbf{R} \subset \mathbf{G}$ and $\mathbf{R} * \mathbf{G}$, $\mathbf{G} * \mathbf{R} \subset \mathbf{B}$ and $\mathbf{B} * \mathbf{G}$, $\mathbf{G} * \mathbf{B} \subset \mathbf{R}$ and
$\mathbf{R}^- * \mathbf{B}$, $\mathbf{B} * \mathbf{R}^-$, $\mathbf{R} * \mathbf{B}^-$, $\mathbf{B}^- * \mathbf{R} \subset \mathbf{G}^-$ and $\mathbf{R}^- * \mathbf{G}$, $\mathbf{G} * \mathbf{R}^-$, $\mathbf{R} * \mathbf{G}^-$, $\mathbf{G}^- * \mathbf{R} \subset \mathbf{B}^-$ and $\mathbf{B}^- * \mathbf{G}$, $\mathbf{G} * \mathbf{B}^-$, $\mathbf{B} * \mathbf{G}^-$, $\mathbf{G}^- * \mathbf{B} \subset \mathbf{R}^-$ and
$\mathbf{R}^- * \mathbf{B}^-$, $\mathbf{B}^- * \mathbf{R}^- \subset \mathbf{G} \oplus \mathbf{G}^-$ and $\mathbf{R}^- * \mathbf{G}^-$, $\mathbf{G}^- * \mathbf{R}^- \subset \mathbf{B} \oplus \mathbf{B}^-$ and $\mathbf{B}^- * \mathbf{G}^-$, $\mathbf{G}^- * \mathbf{B}^- \subset \mathbf{R} \oplus \mathbf{R}^-$

Hence:

If $\boldsymbol{X}'$, $\boldsymbol{Y}'$, $\boldsymbol{Z}' \in \{\mathbf{R}, \mathbf{B}, \mathbf{G}\}$ and $\boldsymbol{X} \in \{\boldsymbol{X}', \boldsymbol{X}'^-\}$, $\boldsymbol{Y} \in \{\boldsymbol{Y}', \boldsymbol{Y}'^-\}$, $\boldsymbol{Z} \in \{\boldsymbol{Z}', \boldsymbol{Z}'^-\}$ then
$\boldsymbol{X} * \boldsymbol{Y} * \boldsymbol{Z} \subset \mathrm{H}$, where again omitting brackets means, that this is valid for any setting of brackets. So, products of any 3 (different) colours (including anti-colours) turn to *WHITE*.

Interaction of roots of unity with spacetime

We are looking for automorphisms on each generation $\mathbb{O}_1$, $\mathbb{O}_2$ and $\mathbb{O}_3$, that are given by interaction with a root of unity $\boldsymbol{a} \coloneqq cos\varphi \boldsymbol{e}_0 + sin\varphi \boldsymbol{z}$, $\boldsymbol{z} \in \mathbb{O}_1, \mathbb{O}_2, \mathbb{O}_3$ respectively, of the form $\boldsymbol{axa}^{-1}$ using:

$\boldsymbol{x} \coloneqq x_0\boldsymbol{e}_0 + x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 + x_4\boldsymbol{e}_4 + x_5\boldsymbol{e}_5 + x_6\boldsymbol{e}_6 + x_7\boldsymbol{e}_7$ and $\boldsymbol{a} \in \mathbb{O}_1$,

$\boldsymbol{x} \coloneqq x_0\boldsymbol{e}_0 + x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 + x_8\boldsymbol{e}_8 + x_9\boldsymbol{e}_9 + x_{10}\boldsymbol{e}_{10} + x_{11}\boldsymbol{e}_{11}$ and $\boldsymbol{a} \in \mathbb{O}_2$,

$\boldsymbol{x} \coloneqq x_0\boldsymbol{e}_0 + x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 + x_{12}\boldsymbol{e}_{12} + x_{13}\boldsymbol{e}_{13} + x_{14}\boldsymbol{e}_{14} + x_{15}\boldsymbol{e}_{15}$ and $\boldsymbol{a} \in \mathbb{O}_3$.

Remember, that $(\boldsymbol{ax})\boldsymbol{a}^{-1} = \boldsymbol{a}(\boldsymbol{xa}^{-1})$ in every Cayley-Dickson algebra.

The following identities hold:

Generation $\mathbb{O}_1$:

$\boldsymbol{z} = \boldsymbol{e}_4$:

$$\boldsymbol{axa}^{-1} = x_0\boldsymbol{e}_0 + (x_1cos2\varphi + x_5sin2\varphi)\boldsymbol{e}_1 + (x_2cos2\varphi + x_6sin2\varphi)\boldsymbol{e}_2 + (x_3cos2\varphi + x_7sin2\varphi)\boldsymbol{e}_3 + x_4\boldsymbol{e}_4 + (x_5cos2\varphi - x_1sin2\varphi)\boldsymbol{e}_5 + (x_6cos2\varphi - x_2sin2\varphi)\boldsymbol{e}_6 + (x_7cos2\varphi - x_3sin2\varphi)\boldsymbol{e}_7$$

$$\boldsymbol{axa} = (x_0cos2\varphi - x_4sin2\varphi)\boldsymbol{e}_0 + x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 + (x_4cos2\varphi + x_0sin2\varphi)\boldsymbol{e}_4 + x_5\boldsymbol{e}_5 + x_6\boldsymbol{e}_6 + x_7\boldsymbol{e}_7$$

So, the operation $\boldsymbol{axa}^{-1}$ leaves coefficients of $\boldsymbol{e}_0$ and $\boldsymbol{e}_4$ unchanged and induces rotations in planes $<\boldsymbol{e}_1, \boldsymbol{e}_5>$, $<\boldsymbol{e}_2, \boldsymbol{e}_6>$ and $<\boldsymbol{e}_3, \boldsymbol{e}_7>$. Whereas $\boldsymbol{axa}$ induces a rotation in $<\boldsymbol{e}_0, \boldsymbol{e}_4>$ and leaving space and anti-space of $\mathbb{O}_1$ unchanged.

$\boldsymbol{z} = \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_5, \boldsymbol{e}_6, \boldsymbol{e}_7$:

$\boldsymbol{e}_1$: $\boldsymbol{axa}^{-1} = x_0\boldsymbol{e}_0 + x_1\boldsymbol{e}_1 + (x_2cos2\varphi - x_3sin2\varphi)\boldsymbol{e}_2 + (x_3cos2\varphi + x_2sin2\varphi)\boldsymbol{e}_3 + (x_4cos2\varphi - x_5sin2\varphi)\boldsymbol{e}_4 + (x_5cos2\varphi + x_4sin2\varphi)\boldsymbol{e}_5 + (x_6cos2\varphi + x_7sin2\varphi)\boldsymbol{e}_6 + (x_7cos2\varphi - x_6sin2\varphi)\boldsymbol{e}_7$

$\boldsymbol{axa} = (x_0cos2\varphi - x_1sin2\varphi)\boldsymbol{e}_0 + (x_1cos2\varphi + x_0sin2\varphi)\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 + x_4\boldsymbol{e}_4 + x_5\boldsymbol{e}_5 + x_6\boldsymbol{e}_6 + x_7\boldsymbol{e}_7$

$\boldsymbol{e}_2$: $\boldsymbol{axa}^{-1} = x_0\boldsymbol{e}_0 + (x_1cos2\varphi + x_3sin2\varphi)\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + (x_3cos2\varphi - x_1sin2\varphi)\boldsymbol{e}_3 + (x_4cos2\varphi - x_6sin2\varphi)\boldsymbol{e}_4 + (x_5cos2\varphi - x_7sin2\varphi)\boldsymbol{e}_5 + (x_6cos2\varphi + x_4sin2\varphi)\boldsymbol{e}_6 + (x_7cos2\varphi + x_5sin2\varphi)\boldsymbol{e}_7$

$\boldsymbol{axa} = (x_0cos2\varphi - x_2sin2\varphi)\boldsymbol{e}_0 + x_1\boldsymbol{e}_1 + (x_2cos2\varphi + x_0sin2\varphi)\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 + x_4\boldsymbol{e}_4 + x_5\boldsymbol{e}_5 + x_6\boldsymbol{e}_6 + x_7\boldsymbol{e}_7$

$\boldsymbol{e}_3$: $\boldsymbol{axa}^{-1} = x_0\boldsymbol{e}_0 + (x_1cos2\varphi - x_2sin2\varphi)\boldsymbol{e}_1 + (x_2cos2\varphi + x_1sin2\varphi)\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 + (x_4cos2\varphi - x_7sin2\varphi)\boldsymbol{e}_4 + (x_5cos2\varphi + x_6sin2\varphi)\boldsymbol{e}_5 + (x_6cos2\varphi - x_5sin2\varphi)\boldsymbol{e}_6 + (x_7cos2\varphi + x_4sin2\varphi)\boldsymbol{e}_7$

$\boldsymbol{axa} = (x_0cos2\varphi - x_3sin2\varphi)\boldsymbol{e}_0 + x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + (x_3cos2\varphi + x_0sin2\varphi)\boldsymbol{e}_3 + x_4\boldsymbol{e}_4 + x_5\boldsymbol{e}_5 + x_6\boldsymbol{e}_6 + x_7\boldsymbol{e}_7$

$\boldsymbol{e}_5$: $\boldsymbol{axa}^{-1} = x_0\boldsymbol{e}_0 + (x_1cos2\varphi - x_4sin2\varphi)\boldsymbol{e}_1 + (x_2cos2\varphi + x_7sin2\varphi)\boldsymbol{e}_2 + (x_3cos2\varphi - x_6sin2\varphi)\boldsymbol{e}_3 + (x_4cos2\varphi + x_1sin2\varphi)\boldsymbol{e}_4 + x_5\boldsymbol{e}_5 + (x_6cos2\varphi + x_3sin2\varphi)\boldsymbol{e}_6 + (x_7cos2\varphi - x_2sin2\varphi)\boldsymbol{e}_7$

$\boldsymbol{axa} = (x_0cos2\varphi - x_5sin2\varphi)\boldsymbol{e}_0 + x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 + x_4\boldsymbol{e}_4 + (x_5cos2\varphi + x_0sin2\varphi)\boldsymbol{e}_5 + x_6\boldsymbol{e}_6 + x_7\boldsymbol{e}_7$

$\boldsymbol{e}_6$: $\boldsymbol{axa}^{-1} = x_0\boldsymbol{e}_0 + (x_1cos2\varphi - x_7sin2\varphi)\boldsymbol{e}_1 + (x_2cos2\varphi - x_4sin2\varphi)\boldsymbol{e}_2 + (x_3cos2\varphi + x_5sin2\varphi)\boldsymbol{e}_3 + (x_4cos2\varphi + x_2sin2\varphi)\boldsymbol{e}_4 + (x_5cos2\varphi - x_3sin2\varphi)\boldsymbol{e}_5 + x_6\boldsymbol{e}_6 + (x_7cos2\varphi + x_1sin2\varphi)\boldsymbol{e}_7$

$\boldsymbol{axa} = (x_0cos2\varphi - x_6sin2\varphi)\boldsymbol{e}_0 + x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 + x_4\boldsymbol{e}_4 + x_5\boldsymbol{e}_5 + (x_6cos2\varphi + x_0sin2\varphi)\boldsymbol{e}_6 + x_7\boldsymbol{e}_7$

$\boldsymbol{e}_7$: $\boldsymbol{axa}^{-1} = x_0\boldsymbol{e}_0 + (x_1cos2\varphi + x_6sin2\varphi)\boldsymbol{e}_1 + (x_2cos2\varphi - x_5sin2\varphi)\boldsymbol{e}_2 + (x_3cos2\varphi - x_4sin2\varphi)\boldsymbol{e}_3 + (x_4cos2\varphi + x_3sin2\varphi)\boldsymbol{e}_4 + (x_5cos2\varphi + x_2sin2\varphi)\boldsymbol{e}_5 + (x_6cos2\varphi - x_1sin2\varphi)\boldsymbol{e}_6 + x_7\boldsymbol{e}_7$

$\boldsymbol{axa} = (x_0cos2\varphi - x_7sin2\varphi)\boldsymbol{e}_0 + x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 + x_4\boldsymbol{e}_4 + x_5\boldsymbol{e}_5 + x_6\boldsymbol{e}_6 + (x_7cos2\varphi + x_0sin2\varphi)\boldsymbol{e}_7$

Generation $\mathbb{O}_2$:

Since the mapping $(\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{e}_6, \boldsymbol{e}_7) \rightarrow (\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_8, \boldsymbol{e}_9, \boldsymbol{e}_{10}, \boldsymbol{e}_{11})$ defines an isomorphism, preserving multiplication of container algebra $\mathbb{S}$, similar identities hold for $\boldsymbol{e}_8$ and $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_9, \boldsymbol{e}_{10}, \boldsymbol{e}_{11}$ of $\mathbb{O}_2$.

Generation $\mathbb{O}_3$:

$\boldsymbol{z} = \boldsymbol{e}_{12}$:

$$\boldsymbol{axa}^{-1} = x_0\boldsymbol{e}_0 + (x_1 cos2\varphi - x_{13} sin2\varphi)\boldsymbol{e}_1 + (x_2 cos2\varphi - x_{14} sin2\varphi)\boldsymbol{e}_2 + (x_3 cos2\varphi - x_{15} sin2\varphi)\boldsymbol{e}_3 + x_{12}\boldsymbol{e}_{12} + (x_{13} cos2\varphi + x_1 sin2\varphi)\boldsymbol{e}_{13} + (x_{14} cos2\varphi + x_2 sin2\varphi)\boldsymbol{e}_{14} + (x_{15} cos2\varphi + x_3 sin2\varphi)\boldsymbol{e}_{15}$$

yellow marked: signs, that differ from similar expression with $\boldsymbol{e}_4$ and $\boldsymbol{e}_8$.

Note, that

$$\boldsymbol{a}^{-1}\boldsymbol{xa} = x_0\boldsymbol{e}_0 + (x_1 cos2\varphi + x_{13} sin2\varphi)\boldsymbol{e}_1 + (x_2 cos2\varphi + x_{14} sin2\varphi)\boldsymbol{e}_2 + (x_3 cos2\varphi + x_{15} sin2\varphi)\boldsymbol{e}_3 + x_{12}\boldsymbol{e}_{12} + (x_{13} cos2\varphi - x_1 sin2\varphi)\boldsymbol{e}_{13} + (x_{14} cos2\varphi - x_2 sin2\varphi)\boldsymbol{e}_{14} + (x_{15} cos2\varphi - x_3 sin2\varphi)\boldsymbol{e}_{15}$$

$\boldsymbol{z} = \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_{13}, \boldsymbol{e}_{14}, \boldsymbol{e}_{15}$:

$\boldsymbol{e}_1$: $$\boldsymbol{axa}^{-1} = x_0\boldsymbol{e}_0 + x_1\boldsymbol{e}_1 + (x_2 cos2\varphi - x_3 sin2\varphi)\boldsymbol{e}_2 + (x_3 cos2\varphi + x_2 sin2\varphi)\boldsymbol{e}_3 + (x_{12} cos2\varphi + x_{13} sin2\varphi)\boldsymbol{e}_{12} + (x_{13} cos2\varphi - x_{12} sin2\varphi)\boldsymbol{e}_{13} + (x_{14} cos2\varphi - x_{15} sin2\varphi)\boldsymbol{e}_{14} + (x_{15} cos2\varphi + x_{14} sin2\varphi)\boldsymbol{e}_{15}$$

yellow marked: signs, that differ from similar expression with $\boldsymbol{e}_1$ in $\mathbb{O}_1$, $\mathbb{O}_2$.

$\boldsymbol{e}_2$: $$\boldsymbol{axa}^{-1} = x_0\boldsymbol{e}_0 + (x_1 cos2\varphi + x_3 sin2\varphi)\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + (x_3 cos2\varphi - x_1 sin2\varphi)\boldsymbol{e}_3 + (x_{12} cos2\varphi + x_{14} sin2\varphi)\boldsymbol{e}_{12} + (x_{13} cos2\varphi + x_{15} sin2\varphi)\boldsymbol{e}_{13} + (x_{14} cos2\varphi - x_{12} sin2\varphi)\boldsymbol{e}_{14} + (x_{15} cos2\varphi - x_{13} sin2\varphi)\boldsymbol{e}_{15}$$

$\boldsymbol{e}_3$: $$\boldsymbol{axa}^{-1} = x_0\boldsymbol{e}_0 + (x_1 cos2\varphi - x_2 sin2\varphi)\boldsymbol{e}_1 + (x_2 cos2\varphi + x_1 sin2\varphi)\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 + (x_{12} cos2\varphi + x_{15} sin2\varphi)\boldsymbol{e}_{12} + (x_{13} cos2\varphi - x_{14} sin2\varphi)\boldsymbol{e}_{13} + (x_{14} cos2\varphi + x_{13} sin2\varphi)\boldsymbol{e}_{14} + (x_{15} cos2\varphi - x_{12} sin2\varphi)\boldsymbol{e}_{15}$$

$\boldsymbol{e}_{13}$: $$\boldsymbol{axa}^{-1} = x_0\boldsymbol{e}_0 + (x_1 cos2\varphi + x_{12} sin2\varphi)\boldsymbol{e}_1 + (x_2 cos2\varphi - x_{15} sin2\varphi)\boldsymbol{e}_2 + (x_3 cos2\varphi + x_{14} sin2\varphi)\boldsymbol{e}_3 + (x_{12} cos2\varphi - x_1 sin2\varphi)\boldsymbol{e}_{12} + x_{13}\boldsymbol{e}_{13} + (x_{14} cos2\varphi - x_3 sin2\varphi)\boldsymbol{e}_{14} + (x_{15} cos2\varphi + x_2 sin2\varphi)\boldsymbol{e}_{15}$$

yellow marked: signs, that differ from similar expression with $\boldsymbol{e}_5$ and $\boldsymbol{e}_9$.

$\boldsymbol{e}_{14}$: $$\boldsymbol{axa}^{-1} = x_0\boldsymbol{e}_0 + (x_1 cos2\varphi + x_{15} sin2\varphi)\boldsymbol{e}_1 + (x_2 cos2\varphi + x_{12} sin2\varphi)\boldsymbol{e}_2 + (x_3 cos2\varphi - x_{13} sin2\varphi)\boldsymbol{e}_3 + (x_{12} cos2\varphi - x_2 sin2\varphi)\boldsymbol{e}_{12} + (x_{13} cos2\varphi + x_3 sin2\varphi)\boldsymbol{e}_{13} + x_{14}\boldsymbol{e}_{14} + (x_{15} cos2\varphi - x_1 sin2\varphi)\boldsymbol{e}_{15}$$

$\boldsymbol{e}_{15}$: $$\boldsymbol{axa}^{-1} = x_0\boldsymbol{e}_0 + (x_1 cos2\varphi - x_{14} sin2\varphi)\boldsymbol{e}_1 + (x_2 cos2\varphi + x_{13} sin2\varphi)\boldsymbol{e}_2 + (x_3 cos2\varphi + x_{12} sin2\varphi)\boldsymbol{e}_3 + (x_{12} cos2\varphi - x_3 sin2\varphi)\boldsymbol{e}_{12} + (x_{13} cos2\varphi - x_2 sin2\varphi)\boldsymbol{e}_{13} + (x_{14} cos2\varphi + x_1 sin2\varphi)\boldsymbol{e}_{14} + x_{15}\boldsymbol{e}_{15}$$

And for $\boldsymbol{axa}$ like for other generations we get:

$$\boldsymbol{axa} = (x_0 cos2\varphi - x_i sin2\varphi)\boldsymbol{e}_0 + (x_i cos2\varphi + x_0 sin2\varphi)\boldsymbol{e}_i + \sum_{j=1, j\neq i}^{7} x_i\boldsymbol{e}_i$$

with i∊{1,2,3,12,13,14,15}

… interpretation to be done …

Crossing of generations

Let us inspect roots of unity crossing the generations. We already marked, that any element $\boldsymbol{a} \in \mathrm{H}$ (subalgebra of $\mathbb{S}$ generated by $\{\boldsymbol{e}_0, \boldsymbol{e}_4, \boldsymbol{e}_8, \boldsymbol{e}_{12}\}$ (isomorphic to $\mathbb{H}$)) satisfies $\boldsymbol{a}(\boldsymbol{a}\boldsymbol{y}) = (\boldsymbol{a}\boldsymbol{a})\boldsymbol{y}$ for every $\boldsymbol{y} \in \mathbb{S}$. Using conjugate mechanism this yields: $\boldsymbol{a}(\boldsymbol{a}\boldsymbol{y}\boldsymbol{a})\boldsymbol{a} = (\boldsymbol{a}\boldsymbol{a})\boldsymbol{y}$ for every $\boldsymbol{y} \in \mathbb{S}$, which implies: $\boldsymbol{a}(\boldsymbol{a}\boldsymbol{y}\boldsymbol{a})\boldsymbol{a} = \boldsymbol{a}^2\boldsymbol{y}\boldsymbol{a}^2$ and $\boldsymbol{a}(\boldsymbol{a}\boldsymbol{y}\boldsymbol{a}^*)\boldsymbol{a}^* = \boldsymbol{a}^2\boldsymbol{y}\boldsymbol{a}^{*2}$. So, if $\boldsymbol{a} \in \mathrm{H}$ is a root of unity, its interaction with spacetime, in fact all of $\mathbb{S}$, shows, that it has a lifecycle and is interacting avoiding zero divisors. Indeed: Since there exists $n \in \mathbb{N}$, such that $\boldsymbol{a}^{2^{n+1}} = 1$, $\boldsymbol{a}\boldsymbol{y} \neq 0$ for every $\boldsymbol{y} \in \mathbb{S}\backslash\{0\}$. Otherwise, the chain $\boldsymbol{a}\boldsymbol{y} \to \boldsymbol{a}(\boldsymbol{a}\boldsymbol{y}) = \boldsymbol{a}^2\boldsymbol{y} \to \boldsymbol{a}^2(\boldsymbol{a}^2\boldsymbol{y}) = \boldsymbol{a}^4\boldsymbol{y} \to \cdots$ yields a contradiction.

Let $\boldsymbol{a} \coloneqq cos\varphi\boldsymbol{e}_0 + sin\varphi\boldsymbol{z} \in \mathrm{H}$ be a root of unity, $\boldsymbol{y} \coloneqq \sum_{i=0}^{15} y_i\boldsymbol{e}_i \in \mathbb{S}$.

$\boldsymbol{z} \coloneqq \frac{1}{\sqrt{2}}(\boldsymbol{e}_4 - \boldsymbol{e}_8)$:

$$\boldsymbol{a}\boldsymbol{y}\boldsymbol{a}^{-1} = \cdots = y_0\boldsymbol{e}_0 + (y_1cos2\varphi + \tfrac{1}{\sqrt{2}}(y_5 - y_9)sin2\varphi)\boldsymbol{e}_1 + (y_2cos2\varphi + \tfrac{1}{\sqrt{2}}(y_6 - y_{10})sin2\varphi)\boldsymbol{e}_2 + (y_3cos2\varphi + \tfrac{1}{\sqrt{2}}(y_7 - y_{11})sin2\varphi)\boldsymbol{e}_3 + (y_4cos^2\varphi - y_8sin^2\varphi - \tfrac{1}{\sqrt{2}}y_{12}sin2\varphi)\boldsymbol{e}_4 + (y_5cos2\varphi - \tfrac{1}{\sqrt{2}}(y_1 + y_{13})sin2\varphi)\boldsymbol{e}_5 + (y_6cos2\varphi - \tfrac{1}{\sqrt{2}}(y_2 + y_{14})sin2\varphi)\boldsymbol{e}_6 + (y_7cos2\varphi - \tfrac{1}{\sqrt{2}}(y_3 + y_{15})sin2\varphi)\boldsymbol{e}_7 + (y_8cos^2\varphi - y_4sin^2\varphi - \tfrac{1}{\sqrt{2}}y_{12}sin2\varphi)\boldsymbol{e}_8 + (y_9cos2\varphi + \tfrac{1}{\sqrt{2}}(y_1 - y_{13})sin2\varphi)\boldsymbol{e}_9 + (y_{10}cos2\varphi + \tfrac{1}{\sqrt{2}}(y_2 - y_{14})sin2\varphi)\boldsymbol{e}_{10} + (y_{11}cos2\varphi + \tfrac{1}{\sqrt{2}}(y_3 - y_{15})sin2\varphi)\boldsymbol{e}_{11} + (y_{12}cos2\varphi + \tfrac{1}{\sqrt{2}}(y_4 + y_8)sin2\varphi)\boldsymbol{e}_{12} + (y_{13}cos2\varphi + \tfrac{1}{\sqrt{2}}(y_5 + y_9)sin2\varphi)\boldsymbol{e}_{13} + (y_{14}cos2\varphi + \tfrac{1}{\sqrt{2}}(y_6 + y_{10})sin2\varphi)\boldsymbol{e}_{14} + (y_{15}cos2\varphi + \tfrac{1}{\sqrt{2}}(y_7 + y_{11})sin2\varphi)\boldsymbol{e}_{15} \eqqcolon \sum_{i=0}^{15} y_i'\boldsymbol{e}_i$$

and we get: $\left|\boldsymbol{a}\boldsymbol{y}\boldsymbol{a}^{-1}\right|^2 = \sum_{i=0}^{15} y_i'^2 = \sum_{i=0}^{15} y_i^2 = |\boldsymbol{y}|^2$. Note, that this is not trivial within sedenions, because of zero divisors in $\mathbb{S}$. For example:

$$(\boldsymbol{e}_1 + \boldsymbol{e}_{12})(\boldsymbol{e}_3 - \boldsymbol{e}_{14}) = 0, \text{ so: } |(\boldsymbol{e}_1 + \boldsymbol{e}_{12})(\boldsymbol{e}_3 - \boldsymbol{e}_{14})|^2 = 0$$

but $|\boldsymbol{e}_1 + \boldsymbol{e}_{12}|^2 * |\boldsymbol{e}_3 - \boldsymbol{e}_{14}|^2 = 4$.

The calculation is lengthy, so let me show norm preservation just for the 3 terms in $\boldsymbol{e}_4$, $\boldsymbol{e}_8$, $\boldsymbol{e}_{12}$ and the 4 terms in $\boldsymbol{e}_1$, $\boldsymbol{e}_5$, $\boldsymbol{e}_9$, $\boldsymbol{e}_{13}$:

$$\begin{aligned}
&(y_4cos^2\varphi - y_8sin^2\varphi - \frac{1}{\sqrt{2}}y_{12}sin2\varphi)^2 + (y_8cos^2\varphi - y_4sin^2\varphi - \frac{1}{\sqrt{2}}y_{12}sin2\varphi)^2 + (y_{12}cos2\varphi + \frac{1}{\sqrt{2}}(y_4 + y_8)sin2\varphi)^2 \\
&= y_{12}^2 + y_4^2cos^4\varphi + y_8^2sin^4\varphi + y_8^2cos^4\varphi + y_4^2sin^4\varphi + \frac{1}{2}y_4^2sin^22\varphi + \frac{1}{2}y_8^2sin^22\varphi + y_4y_8sin^22\varphi - 4y_4y_8cos^2\varphi sin^2\varphi \\
&= y_{12}^2 + y_4^2(cos^4\varphi + sin^4\varphi + 2cos^2\varphi sin^2\varphi) + y_8^2(cos^4\varphi + sin^4\varphi + 2cos^2\varphi sin^2\varphi) = y_{12}^2 + y_4^2 + y_8^2
\end{aligned}$$

$$\begin{aligned}
&(y_1cos2\varphi + \frac{1}{\sqrt{2}}(y_5 - y_9)sin2\varphi)^2 + (y_5cos2\varphi - \frac{1}{\sqrt{2}}(y_1 + y_{13})sin2\varphi)^2 + (y_9cos2\varphi + \frac{1}{\sqrt{2}}(y_1 - y_{13})sin2\varphi)^2 + (y_{13}cos2\varphi + \frac{1}{\sqrt{2}}(y_5 + y_9)sin2\varphi)^2 \\
&= y_1^2(cos^22\varphi + sin^22\varphi) + y_5^2(cos^22\varphi + sin^22\varphi) + y_9^2(cos^22\varphi + sin^22\varphi) + y_{13}^2(cos^22\varphi + sin^22\varphi) = y_1^2 + y_5^2 + y_9^2 + y_{13}^2
\end{aligned}$$

Restriction to generation I:

$\boldsymbol{x} \coloneqq \sum_{i=0}^{3} x_i\boldsymbol{e}_i \in \mathbb{H}$:

$\boldsymbol{axa^{-1}} = x_0\boldsymbol{e}_0 + x_1 cos2\varphi\boldsymbol{e}_1 + x_2 cos2\varphi\boldsymbol{e}_2 + x_3 cos2\varphi\boldsymbol{e}_3 - \frac{1}{\sqrt{2}}x_1 sin2\varphi\boldsymbol{e}_5 - \frac{1}{\sqrt{2}}x_2 sin2\varphi\boldsymbol{e}_6 - \frac{1}{\sqrt{2}}x_3 sin2\varphi\boldsymbol{e}_7 + \frac{1}{\sqrt{2}}x_1 sin2\varphi\boldsymbol{e}_9 + \frac{1}{\sqrt{2}}x_2 sin2\varphi\boldsymbol{e}_{10} + \frac{1}{\sqrt{2}}x_3 sin2\varphi\boldsymbol{e}_{11} =$
$x_0\boldsymbol{e}_0 + cos2\varphi(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3) - sin2\varphi\frac{1}{\sqrt{2}}(x_1(\boldsymbol{e}_5 - \boldsymbol{e}_9) + x_2(\boldsymbol{e}_6 - \boldsymbol{e}_{10}) + x_3(\boldsymbol{e}_7 - \boldsymbol{e}_{11}))$

Note, that **all** 3 terms $(\boldsymbol{e}_5 - \boldsymbol{e}_9), (\boldsymbol{e}_6 - \boldsymbol{e}_{10}), (\boldsymbol{e}_7 - \boldsymbol{e}_{11})$ are zero divisors. Indeed:
$(\boldsymbol{e}_6 + \boldsymbol{e}_{10})(\boldsymbol{e}_5 - \boldsymbol{e}_9) = (\boldsymbol{e}_5 + \boldsymbol{e}_9)(\boldsymbol{e}_6 - \boldsymbol{e}_{10}) = (\boldsymbol{e}_5 + \boldsymbol{e}_9)(\boldsymbol{e}_7 - \boldsymbol{e}_{11}) = 0$

$\boldsymbol{z} := \frac{1}{\sqrt{2}}(\boldsymbol{e}_8 - \boldsymbol{e}_{12})$:

$\boldsymbol{aya^{-1}} = \cdots = y_0\boldsymbol{e}_0 + (y_1 cos2\varphi + \frac{1}{\sqrt{2}}(y_9 + y_{13})sin2\varphi)\boldsymbol{e}_1 + (y_2 cos2\varphi + \frac{1}{\sqrt{2}}(y_{10} + y_{14})sin2\varphi)\boldsymbol{e}_2 + (y_3 cos2\varphi + \frac{1}{\sqrt{2}}(y_{11} + y_{15})sin2\varphi)\boldsymbol{e}_3 + (y_4 cos2\varphi + \frac{1}{\sqrt{2}}(y_8 + y_{12})sin2\varphi)\boldsymbol{e}_4 + (y_5 cos2\varphi + \frac{1}{\sqrt{2}}(y_{13} - y_9)sin2\varphi)\boldsymbol{e}_5 + (y_6 cos2\varphi + \frac{1}{\sqrt{2}}(y_{14} - y_{10})sin2\varphi)\boldsymbol{e}_6 + (y_7 cos2\varphi + \frac{1}{\sqrt{2}}(y_{15} - y_{11})sin2\varphi)\boldsymbol{e}_7 + (y_8 cos^2\varphi - y_{12} sin^2\varphi - \frac{1}{\sqrt{2}}y_4 sin2\varphi)\boldsymbol{e}_8 + (y_9 cos2\varphi + \frac{1}{\sqrt{2}}(y_5 - y_1)sin2\varphi)\boldsymbol{e}_9 + (y_{10} cos2\varphi + \frac{1}{\sqrt{2}}(y_6 - y_2)sin2\varphi)\boldsymbol{e}_{10} + (y_{11} cos2\varphi + \frac{1}{\sqrt{2}}(y_7 - y_3)sin2\varphi)\boldsymbol{e}_{11} + (y_{12} cos2\varphi - y_8 sin^2\varphi - \frac{1}{\sqrt{2}}y_4 sin2\varphi)\boldsymbol{e}_{12} + (y_{13} cos2\varphi - \frac{1}{\sqrt{2}}(y_1 + y_5)sin2\varphi)\boldsymbol{e}_{13} + (y_{14} cos2\varphi - \frac{1}{\sqrt{2}}(y_2 + y_6)sin2\varphi)\boldsymbol{e}_{14} + (y_{15} cos2\varphi - \frac{1}{\sqrt{2}}(y_3 + y_7)sin2\varphi)\boldsymbol{e}_{15} =: \sum_{i=0}^{15} y_i{}'\boldsymbol{e}_i$

and just like above we get: $|\boldsymbol{aya^{-1}}|^2 = \sum_{i=0}^{15} y_i{}'^2 = \sum_{i=0}^{15} y_i{}^2 = |\boldsymbol{y}|^2$.

Restriction to generation I:

$\boldsymbol{x} := \sum_{i=0}^{3} x_i\boldsymbol{e}_i \in \mathbb{H}$:

$\boldsymbol{axa^{-1}} = x_0\boldsymbol{e}_0 + x_1 cos2\varphi\boldsymbol{e}_1 + x_2 cos2\varphi\boldsymbol{e}_2 + x_3 cos2\varphi\boldsymbol{e}_3 - \frac{1}{\sqrt{2}}x_1 sin2\varphi\boldsymbol{e}_9 - \frac{1}{\sqrt{2}}x_2 sin2\varphi\boldsymbol{e}_{10} - \frac{1}{\sqrt{2}}x_3 sin2\varphi\boldsymbol{e}_{11} - \frac{1}{\sqrt{2}}x_1 sin2\varphi\boldsymbol{e}_{13} - \frac{1}{\sqrt{2}}x_2 sin2\varphi\boldsymbol{e}_{14} - \frac{1}{\sqrt{2}}x_3 sin2\varphi\boldsymbol{e}_{15} =$
$x_0\boldsymbol{e}_0 + cos2\varphi(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3) - sin2\varphi\frac{1}{\sqrt{2}}(x_1(\boldsymbol{e}_9 + \boldsymbol{e}_{13}) + x_2(\boldsymbol{e}_{10} + \boldsymbol{e}_{14}) + x_3(\boldsymbol{e}_{11} + \boldsymbol{e}_{15}))$

Note, that **none** of 3 terms $(\boldsymbol{e}_9 + \boldsymbol{e}_{13}), (\boldsymbol{e}_{10} + \boldsymbol{e}_{14}), (\boldsymbol{e}_{11} + \boldsymbol{e}_{15})$ is a zero divisor in $\mathbb{S}$.

$\boldsymbol{z} := \frac{1}{\sqrt{2}}(\boldsymbol{e}_{12} - \boldsymbol{e}_4)$:

$\boldsymbol{aya^{-1}} = \cdots = y_0\boldsymbol{e}_0 + (y_1 cos2\varphi - \frac{1}{\sqrt{2}}(y_5 + y_{13})sin2\varphi)\boldsymbol{e}_1 + (y_2 cos2\varphi - \frac{1}{\sqrt{2}}(y_6 + y_{14})sin2\varphi)\boldsymbol{e}_2 + (y_3 cos2\varphi - \frac{1}{\sqrt{2}}(y_7 + y_{15})sin2\varphi)\boldsymbol{e}_3 + (y_4 cos^2\varphi - y_{12} sin^2\varphi - \frac{1}{\sqrt{2}}y_8 sin2\varphi)\boldsymbol{e}_4 + (y_5 cos2\varphi + \frac{1}{\sqrt{2}}(y_1 + y_9)sin2\varphi)\boldsymbol{e}_5 + (y_6 cos2\varphi + \frac{1}{\sqrt{2}}(y_2 + y_{10})sin2\varphi)\boldsymbol{e}_6 + (y_7 cos2\varphi + \frac{1}{\sqrt{2}}(y_3 + y_{11})sin2\varphi)\boldsymbol{e}_7 + (y_8 cos2\varphi + \frac{1}{\sqrt{2}}(y_4 + y_{12})sin2\varphi)\boldsymbol{e}_8 + (y_9 cos2\varphi - \frac{1}{\sqrt{2}}(y_5 - y_{13})sin2\varphi)\boldsymbol{e}_9 + (y_{10} cos2\varphi - \frac{1}{\sqrt{2}}(y_6 - y_{14})sin2\varphi)\boldsymbol{e}_{10} + (y_{11} cos2\varphi - \frac{1}{\sqrt{2}}(y_7 - y_{15})sin2\varphi)\boldsymbol{e}_{11} + (y_{12} cos^2\varphi - y_4 sin^2\varphi - \frac{1}{\sqrt{2}}y_8 sin2\varphi)\boldsymbol{e}_{12} + (y_{13} cos2\varphi + \frac{1}{\sqrt{2}}(y_1 - y_9)sin2\varphi)\boldsymbol{e}_{13} + (y_{14} cos2\varphi + \frac{1}{\sqrt{2}}(y_2 - y_{10})sin2\varphi)\boldsymbol{e}_{14} + (y_{15} cos2\varphi + \frac{1}{\sqrt{2}}(y_3 - y_{11})sin2\varphi)\boldsymbol{e}_{15} =: \sum_{i=0}^{15} y_i{}'\boldsymbol{e}_i$

and just like above we get: $|\boldsymbol{aya^{-1}}|^2 = \sum_{i=0}^{15} y_i{}'^2 = \sum_{i=0}^{15} y_i{}^2 = |\boldsymbol{y}|^2$.

Restriction to generation I:

$\boldsymbol{x} \coloneqq \sum_{i=0}^{3} x_i \boldsymbol{e}_i \in \mathbb{H}$:

$\boldsymbol{axa^{-1}} = x_0\boldsymbol{e}_0 + x_1 cos2\varphi \boldsymbol{e}_1 + x_2 cos2\varphi \boldsymbol{e}_2 + x_3 cos2\varphi \boldsymbol{e}_3 + \frac{1}{\sqrt{2}} x_1 sin2\varphi \boldsymbol{e}_5 + \frac{1}{\sqrt{2}} x_2 sin2\varphi \boldsymbol{e}_6 + \frac{1}{\sqrt{2}} x_3 sin2\varphi \boldsymbol{e}_7 + \frac{1}{\sqrt{2}} x_1 sin2\varphi \boldsymbol{e}_{13} + \frac{1}{\sqrt{2}} x_2 sin2\varphi \boldsymbol{e}_{14} + \frac{1}{\sqrt{2}} x_3 sin2\varphi \boldsymbol{e}_{15} =$
$x_0\boldsymbol{e}_0 + cos2\varphi(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3) + sin2\varphi \frac{1}{\sqrt{2}}(x_1(\boldsymbol{e}_5 + \boldsymbol{e}_{13}) + x_2(\boldsymbol{e}_6 + \boldsymbol{e}_{14}) + x_3(\boldsymbol{e}_7 + \boldsymbol{e}_{15}))$

Note, that **none** of 3 terms $(\boldsymbol{e}_5 + \boldsymbol{e}_{13}), (\boldsymbol{e}_6 + \boldsymbol{e}_{14}), (\boldsymbol{e}_7 + \boldsymbol{e}_{15})$ is a zero divisor in $\mathbb{S}$. This and the absence of $\boldsymbol{e}_4$ (associated with electron) in the root of unity will lead to the setting of the $\boldsymbol{e}$-neutrino to be that root of unity. It will enable the neutron to be defined as root of unity based on a sum of proton ($\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $-\boldsymbol{e}_3$), electron (-$\boldsymbol{e}_4$) and anti-e-neutrino ($\boldsymbol{e}_8$,$-\boldsymbol{e}_{12}$).

$\boldsymbol{z} \coloneqq \frac{1}{\sqrt{6}}(\boldsymbol{e}_4 + \boldsymbol{e}_8 - 2\boldsymbol{e}_{12})$:

$\boldsymbol{aya^{-1}} = \cdots = y_0\boldsymbol{e}_0 + (y_1 cos2\varphi + \frac{1}{\sqrt{6}}(y_5 + y_9 + 2y_{13}) sin2\varphi)\boldsymbol{e}_1 + (y_2 cos2\varphi + \frac{1}{\sqrt{6}}(y_6 + y_{10} + 2y_{14}) sin2\varphi)\boldsymbol{e}_2 + (y_3 cos2\varphi + \frac{1}{\sqrt{6}}(y_7 + y_{11} + 2y_{15}) sin2\varphi)\boldsymbol{e}_3 + (y_4 cos^2\varphi - \frac{1}{6} sin^2\varphi(4y_4 - 2y_8 + 4y_{12}) + \frac{1}{\sqrt{6}}(2y_8 + y_{12}) sin2\varphi)\boldsymbol{e}_4 + (y_5 cos2\varphi + \frac{1}{\sqrt{6}}(-y_1 - 2y_9 + y_{13}) sin2\varphi)\boldsymbol{e}_5 + (y_6 cos2\varphi + \frac{1}{\sqrt{6}}(-y_2 - 2y_{10} + y_{14}) sin2\varphi)\boldsymbol{e}_6 + (y_7 cos2\varphi + \frac{1}{\sqrt{6}}(-y_3 - 2y_{11} + y_{15}) sin2\varphi)\boldsymbol{e}_7 + (y_8 cos^2\varphi + \frac{1}{6} sin^2\varphi(2y_4 - 4y_8 - 4y_{12}) + \frac{1}{\sqrt{6}}(-2y_4 - y_{12}) sin2\varphi)\boldsymbol{e}_8 + (y_9 cos2\varphi + \frac{1}{\sqrt{6}}(-y_1 + 2y_5 - y_{13}) sin2\varphi)\boldsymbol{e}_9 + (y_{10} cos2\varphi + \frac{1}{\sqrt{6}}(-y_2 + 2y_6 - y_{14}) sin2\varphi)\boldsymbol{e}_{10} + (y_{11} cos2\varphi + \frac{1}{\sqrt{6}}(-y_3 + 2y_7 - y_{15}) sin2\varphi)\boldsymbol{e}_{11} + (y_{12} cos^2\varphi + \frac{1}{6} sin^2\varphi(-4y_4 - 4y_8 + 2y_{12}) + \frac{1}{\sqrt{6}}(-y_4 + y_8) sin2\varphi)\boldsymbol{e}_{12} + (y_{13} cos2\varphi + \frac{1}{\sqrt{6}}(-2y_1 - y_5 + y_9) sin2\varphi)\boldsymbol{e}_{13} + (y_{14} cos2\varphi + \frac{1}{\sqrt{6}}(-2y_2 - y_6 + y_{10}) sin2\varphi)\boldsymbol{e}_{14} + (y_{15} cos2\varphi + \frac{1}{\sqrt{6}}(-2y_3 - y_7 + y_{11}) sin2\varphi)\boldsymbol{e}_{15} \eqqcolon \sum_{i=0}^{15} y_i' \boldsymbol{e}_i$

and just like above we get: $|\boldsymbol{aya}^{-1}|^2 = \sum_{i=0}^{15} {y_i'}^2 = \sum_{i=0}^{15} y_i^2 = |\boldsymbol{y}|^2$.

The calculation is lengthy, so let me show norm preservation just for the 3 terms in $\boldsymbol{e}_4$, $\boldsymbol{e}_8$, $\boldsymbol{e}_{12}$:

$$(y_4 cos^2\varphi - \frac{1}{6} sin^2\varphi(4y_4 - 2y_8 + 4y_{12}) + \frac{1}{\sqrt{6}}(2y_8 + y_{12}) sin2\varphi)^2 + (y_8 cos^2\varphi + \frac{1}{6} sin^2\varphi(2y_4 - 4y_8 - 4y_{12}) + \frac{1}{\sqrt{6}}(-2y_4 - y_{12}) sin2\varphi)^2 + (y_{12} cos^2\varphi + \frac{1}{6} sin^2\varphi(-4y_4 - 4y_8 + 2y_{12}) + \frac{1}{\sqrt{6}}(-y_4 + y_8) sin2\varphi)^2 = \cdots$$
$$= ({y_4}^2 + {y_8}^2 + {y_{12}}^2) cos^4\varphi + \frac{1}{6} cos^2\varphi sin^2 2\varphi(12{y_4}^2 + 12{y_8}^2 + 12{y_{12}}^2) + sin^4\varphi({y_4}^2 + {y_8}^2 + {y_{12}}^2) = {y_4}^2 + {y_8}^2 + {y_{12}}^2$$

Restriction to generation I:

$\boldsymbol{x} \coloneqq \sum_{i=0}^{3} x_i \boldsymbol{e}_i \in \mathbb{H}$:

$\boldsymbol{axa^{-1}} = x_0\boldsymbol{e}_0 + x_1 cos2\varphi \boldsymbol{e}_1 + x_2 cos2\varphi \boldsymbol{e}_2 + x_3 cos2\varphi \boldsymbol{e}_3 - \frac{1}{\sqrt{6}} x_1 sin2\varphi \boldsymbol{e}_5 - \frac{1}{\sqrt{6}} x_2 sin2\varphi \boldsymbol{e}_6 - \frac{1}{\sqrt{6}} x_3 sin2\varphi \boldsymbol{e}_7 - \frac{1}{\sqrt{6}} x_1 sin2\varphi \boldsymbol{e}_9 - \frac{1}{\sqrt{6}} x_2 sin2\varphi \boldsymbol{e}_{10} - \frac{1}{\sqrt{6}} x_3 sin2\varphi \boldsymbol{e}_{11} - \frac{2}{\sqrt{6}} x_1 sin2\varphi \boldsymbol{e}_{13} - \frac{2}{\sqrt{6}} x_2 sin2\varphi \boldsymbol{e}_{14} - \frac{2}{\sqrt{6}} x_3 sin2\varphi \boldsymbol{e}_{15} =$

$x_0\boldsymbol{e}_0 + cos2\varphi(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3) - sin2\varphi\frac{1}{\sqrt{6}}(x_1(\boldsymbol{e}_5 + \boldsymbol{e}_9 + 2\boldsymbol{e}_{13}) + x_2(\boldsymbol{e}_6 + \boldsymbol{e}_{10} + 2\boldsymbol{e}_{14}) + x_3(\boldsymbol{e}_7 + \boldsymbol{e}_{11} + 2\boldsymbol{e}_{15}))$

… interpretation to be done …

Elementary particles

Where are the elementary particles?

Notation: If we omit brackets in a product of elements of *sedenions*, the meaning shall be, that whatever is claimed to be true, is true for any setting of brackets.

Electron and positron

Electron: members of *summed roots of unity* with one $\boldsymbol{e}_i$ of one generation ($\mathbb{O}_1$, $\mathbb{O}_2$ or $\mathbb{O}_3$) which is colour white, i.e. $\boldsymbol{e}_i \in \{\boldsymbol{e}_4, \boldsymbol{e}_8, \boldsymbol{e}_{12}\}$. So, with $\boldsymbol{z} \coloneqq \pm\boldsymbol{e}_4$ $(\boldsymbol{e})$ / $\boldsymbol{z} \coloneqq \pm\boldsymbol{e}_8$ $(\boldsymbol{\mu})$ / $\boldsymbol{z} \coloneqq \pm\boldsymbol{e}_{12}$ $(\boldsymbol{\tau})$ and $\varphi = (2m-1)\frac{\pi}{2^n}$, m∈{0,…,2^{n-2}} (1<n∈ℕ), the element $\boldsymbol{a} \coloneqq cos\varphi\boldsymbol{e}_0 + sin\varphi\boldsymbol{z}$ is a root of unity and is element of $\mathbb{O}_1$ / $\mathbb{O}_2$ / $\mathbb{O}_3$ (generation I / II / III) and $\boldsymbol{e}_i \in \mathbb{H}$ (White, $\boldsymbol{e}_i$ is colour neutral).

$\boldsymbol{z} \coloneqq \boldsymbol{e}_4$ $(\boldsymbol{e}^+)$ yields the positron, $-\boldsymbol{e}_4$ $(\boldsymbol{e}^-)$ yields the electron as associated root of unity:

$\boldsymbol{y} \coloneqq \boldsymbol{axa}^{-1} = x_0\boldsymbol{e}_0 + (x_1cos2\varphi + x_5sin2\varphi)\boldsymbol{e}_1 + (x_2cos2\varphi + x_6sin2\varphi)\boldsymbol{e}_2 + (x_3cos2\varphi + x_7sin2\varphi)\boldsymbol{e}_3 + x_4\boldsymbol{e}_4 + (x_5cos2\varphi - x_1sin2\varphi)\boldsymbol{e}_5 + (x_6cos2\varphi - x_2sin2\varphi)\boldsymbol{e}_6 + (x_7cos2\varphi - x_3sin2\varphi)\boldsymbol{e}_7$

and $\boldsymbol{yy}^* = {x_0}^2 + {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 + {x_5}^2 + {x_6}^2 + {x_7}^2$

$\boldsymbol{z} \coloneqq \boldsymbol{e}_8$ $(\boldsymbol{\mu}^+)$, $-\boldsymbol{e}_8$ $(\boldsymbol{\mu}^-)$:

$\boldsymbol{y} \coloneqq \boldsymbol{axa}^{-1} = x_0\boldsymbol{e}_0 + (x_1cos2\varphi + x_5sin2\varphi)\boldsymbol{e}_1 + (x_2cos2\varphi + x_6sin2\varphi)\boldsymbol{e}_2 + (x_3cos2\varphi + x_7sin2\varphi)\boldsymbol{e}_3 + x_8\boldsymbol{e}_8 + (x_9cos2\varphi - x_1sin2\varphi)\boldsymbol{e}_9 + (x_{10}cos2\varphi - x_2sin2\varphi)\boldsymbol{e}_{10} + (x_{11}cos2\varphi - x_3sin2\varphi)\boldsymbol{e}_{11}$

and $\boldsymbol{yy}^* = {x_0}^2 + {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_8}^2 + {x_9}^2 + {x_{10}}^2 + {x_{11}}^2$

$\boldsymbol{z} \coloneqq \boldsymbol{e}_{12}$ $(\boldsymbol{\tau}^+)$, $-\boldsymbol{e}_{12}$ $(\boldsymbol{\tau}^-)$:

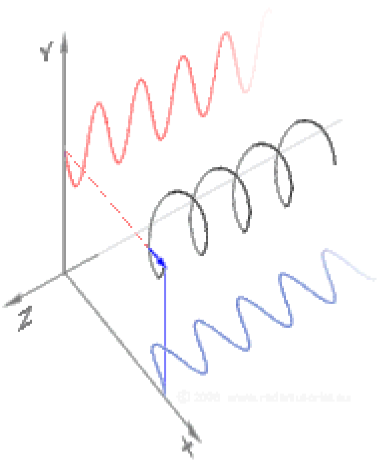
$\boldsymbol{y} \coloneqq \boldsymbol{axa}^{-1} = x_0\boldsymbol{e}_0 + (x_1cos2\varphi - x_{13}sin2\varphi)\boldsymbol{e}_1 + (x_2cos2\varphi - x_{14}sin2\varphi)\boldsymbol{e}_2 + (x_3cos2\varphi - x_{15}sin2\varphi)\boldsymbol{e}_3 + x_{12}\boldsymbol{e}_{12} + (x_{13}cos2\varphi + x_1sin2\varphi)\boldsymbol{e}_{13} + (x_{14}cos2\varphi + x_2sin2\varphi)\boldsymbol{e}_{14} + (x_{15}cos2\varphi + x_3sin2\varphi)\boldsymbol{e}_{15}$

and $\boldsymbol{yy}^* = {x_0}^2 + {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_{12}}^2 + {x_{13}}^2 + {x_{114}}^2 + {x_{15}}^2$

Note the yellow marked difference in signs for generation III.

If we associate δx_0 with change in time, $\delta x_1\boldsymbol{e}_1 + \delta x_2\boldsymbol{e}_2 + \delta x_3\boldsymbol{e}_3$ with change in space and $\delta x_1\boldsymbol{e}_5 + \delta x_2\boldsymbol{e}_6 + \delta x_3\boldsymbol{e}_7$ with change in »anti-space«, incrementing of $2^{i+1}\varphi$ induces kind of a wave on space having direction $\frac{\delta x}{\|\delta x\|}$ and a wave with a phase shift of $\frac{\pi}{2}$ on anti-space.

The following picture from WIKIPEDIA (https://de.wikipedia.org/wiki/Polarisation) illustrates the situation



In the chapter »On mass and charge« to come we will see, that our view on the electron scales the axes of the unit circle associated with the root of unity in such an extreme way, that the resulting ellipse has a quotient of axes of magnitude 10^{45}. This makes the electron from an outer view look like a line rather than an ellipse. In the above picture this means that we have the impression of a linear polarization. The plane of polarization is spanned by direction $\frac{\delta x}{\|\delta x\|}$ and a line in space that corresponds outside to inner axe time.

Photon

We can define a photon as an operation of scaled multiplication of the form:

$(\boldsymbol{e}^{+})^{i}\lambda\frac{\delta x}{\|\delta x\|}(\boldsymbol{e}^{-})^{i}$ with real positive λ and i∈ℕ, λ being the wavelength.

Quarks

The pair $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ generates the algebra $\mathbb{H}$ $(\boldsymbol{e}_3 = \boldsymbol{e}_1 * \boldsymbol{e}_2)$. For the definition of a quark set $\boldsymbol{z} \coloneqq \boldsymbol{e}_1$. So, in contrast to standard model, the charge of the quark is 1. But it appears within the proton (root of unity with $\boldsymbol{z} \coloneqq \frac{2}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 - \frac{1}{3}\boldsymbol{e}_3$) as $\frac{2}{3}\boldsymbol{e}_1$, that is with a charge of $\frac{2}{3}$. The idea is, that we only see the "proton-quark". Moreover, what is an up-or down-quark shall be defined by the charge of its use in a summed root of unity. For example, in $\boldsymbol{z} \coloneqq \frac{2}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 - \frac{1}{3}\boldsymbol{e}_3$ the parts $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ do not differ.

In the model presented here quarks of different generations differ from one another not by their definition as some root of unity but by their action on one of the three subalgebras $\mathbb{O}_1$, $\mathbb{O}_2$ and $\mathbb{O}_3$ of $\mathbb{S}$. Let us inspect the interaction of the root of unity $\boldsymbol{q} \coloneqq cos\varphi\boldsymbol{e}_0 + sin\varphi\boldsymbol{e}_1$ with the 3 subalgebras.

Generation I:

$\boldsymbol{y} \coloneqq \boldsymbol{qxq}^{-1} = x_0\boldsymbol{e}_0 + x_1\boldsymbol{e}_1 + (x_2cos2\varphi - x_3sin2\varphi)\boldsymbol{e}_2 + (x_3cos2\varphi + x_2sin2\varphi)\boldsymbol{e}_3 + (x_4cos2\varphi - x_5sin2\varphi)\boldsymbol{e}_4 + (x_5cos2\varphi + x_4sin2\varphi)\boldsymbol{e}_5 + (x_6cos2\varphi + x_7sin2\varphi)\boldsymbol{e}_6 + (x_7cos2\varphi - x_6sin2\varphi)\boldsymbol{e}_7$

and $\boldsymbol{yy}^{*} = x_0{}^2 + x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2 + x_5{}^2 + x_6{}^2 + x_7{}^2$

One gets similar results for $\boldsymbol{q} \coloneqq cos\varphi\boldsymbol{e}_0 + sin\varphi\boldsymbol{e}_2$ and $\boldsymbol{q} \coloneqq cos\varphi\boldsymbol{e}_0 + sin\varphi\boldsymbol{e}_3$.

If we restrict to spacetime, this defines a rotation in the plane generated by $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$.

Since $\boldsymbol{q}(\delta\boldsymbol{x})^2\boldsymbol{q}^* = \boldsymbol{q}\delta\boldsymbol{x}(\boldsymbol{q}^*\boldsymbol{q})\delta\boldsymbol{x}\boldsymbol{q}^* = (\boldsymbol{q}\delta\boldsymbol{x}\boldsymbol{q}^*)(\boldsymbol{q}\delta\boldsymbol{x}\boldsymbol{q}^*)$ we have a transformation preserving the Minkowski metric, i.e. an element of G_{15}, namely a rotation.

Generation II:

$\boldsymbol{y} \coloneqq \boldsymbol{q}\boldsymbol{x}\boldsymbol{q}^{-1} = x_0\boldsymbol{e}_0 + x_1\boldsymbol{e}_1 + (x_2cos2\varphi - x_3sin2\varphi)\boldsymbol{e}_2 + (x_3cos2\varphi + x_2sin2\varphi)\boldsymbol{e}_3 + (x_8cos2\varphi - x_9sin2\varphi)\boldsymbol{e}_8 + (x_9cos2\varphi + x_8sin2\varphi)\boldsymbol{e}_9 + (x_{10}cos2\varphi + x_{11}sin2\varphi)\boldsymbol{e}_{10} + (x_{11}cos2\varphi - x_{10}sin2\varphi)\boldsymbol{e}_{11}$

and $\boldsymbol{y}\boldsymbol{y}^* = {x_0}^2 + {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_8}^2 + {x_9}^2 + {x_{10}}^2 + {x_{11}}^2$

One gets similar results for $\boldsymbol{q} \coloneqq cos\varphi\boldsymbol{e}_0 + sin\varphi\boldsymbol{e}_2$ and $\boldsymbol{q} \coloneqq cos\varphi\boldsymbol{e}_0 + sin\varphi\boldsymbol{e}_3$.

Generation III:

$\boldsymbol{y} \coloneqq \boldsymbol{q}\boldsymbol{x}\boldsymbol{q}^{-1} = x_0\boldsymbol{e}_0 + x_1\boldsymbol{e}_1 + (x_2cos2\varphi - x_3sin2\varphi)\boldsymbol{e}_2 + (x_3cos2\varphi + x_2sin2\varphi)\boldsymbol{e}_3 + (x_{12}cos2\varphi + x_{13}sin2\varphi)\boldsymbol{e}_{12} + (x_{13}cos2\varphi - x_{12}sin2\varphi)\boldsymbol{e}_{13} + (x_{14}cos2\varphi - x_{15}sin2\varphi)\boldsymbol{e}_{14} + (x_{15}cos2\varphi + x_{14}sin2\varphi)\boldsymbol{e}_{15}$

and $\boldsymbol{y}\boldsymbol{y}^* = {x_0}^2 + {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_{12}}^2 + {x_{13}}^2 + {x_{14}}^2 + {x_{15}}^2$

One gets similar results for $\boldsymbol{q} \coloneqq cos\varphi\boldsymbol{e}_0 + sin\varphi\boldsymbol{e}_2$ and $\boldsymbol{q} \coloneqq cos\varphi\boldsymbol{e}_0 + sin\varphi\boldsymbol{e}_3$.

Note the (yellow marked) differences in sign of generations I/II and generation III. This has to do with the fact, that $\mathbb{O}_3$ is not isomorphic to $\mathbb{O}$, in contrast to $\mathbb{O}_1$, $\mathbb{O}_2$.

Gluon

With $\boldsymbol{z} \coloneqq \frac{1}{\sqrt{6}}(2\boldsymbol{e}_4 - \boldsymbol{e}_8 - \boldsymbol{e}_{12})$ and $\varphi = (2m-1)\frac{\pi}{2^n}$, $m\in\{0,\ldots,2^{n-2}\}$ $(1<n\in\mathbb{N})$ set $\boldsymbol{g} \coloneqq cos\varphi\boldsymbol{e}_0 + sin\varphi\boldsymbol{z}$. Then its interaction with spacetime/anti-spacetime looks like:

… work to be done …

Neutrinos

Generation I: Set $\boldsymbol{z} \coloneqq \frac{1}{\sqrt{2}}(\boldsymbol{e}_{12} - \boldsymbol{e}_8)$ and $\varphi = (2m-1)\frac{\pi}{2^n}$, $m\in\{0,\ldots,2^{n-2}\}$ $(1<n\in\mathbb{N})$, and define $\boldsymbol{\nu}_e \coloneqq cos\varphi\boldsymbol{e}_0 + sin\varphi\boldsymbol{z}$.

Generation II: Set $\boldsymbol{z} \coloneqq \frac{1}{\sqrt{2}}(\boldsymbol{e}_4 - \boldsymbol{e}_{12})$ and $\varphi = (2m-1)\frac{\pi}{2^n}$, $m\in\{0,\ldots,2^{n-2}\}$ $(1<n\in\mathbb{N})$, and define $\boldsymbol{\nu}_\mu \coloneqq cos\varphi\boldsymbol{e}_0 + sin\varphi\boldsymbol{z}$.

Generation III: Set $\boldsymbol{z} \coloneqq \frac{1}{\sqrt{2}}(\boldsymbol{e}_8 - \boldsymbol{e}_4)$ and $\varphi = (2m-1)\frac{\pi}{2^n}$, $m\in\{0,\ldots,2^{n-2}\}$ $(1<n\in\mathbb{N})$, and define $\boldsymbol{\nu}_\tau \coloneqq cos\varphi\boldsymbol{e}_0 + sin\varphi\boldsymbol{z}$.

Remember, that interaction restricted to spacetime for $\boldsymbol{\nu}_\tau$ of generation III yields terms that are zero divisors within $\mathbb{S}$, whereas this is not the case for $\boldsymbol{\nu}_e$ and $\boldsymbol{\nu}_\mu$. For $\boldsymbol{\nu}_e$ mass is given by our view on the vector $\boldsymbol{z} \coloneqq \frac{1}{\sqrt{2}}(\boldsymbol{e}_{12} - \boldsymbol{e}_8)$, which is completely outside $\mathbb{O}_1$. So, from a mere $\mathbb{O}_1$-perspective $\boldsymbol{\nu}_e$ will look like massless.

… work to be done …

Neutrino-type black holes

All the instances of roots of unity of a given type must have different locations in spacetime (defined up to an uncertainty given by Heisenberg's principle). For those types with sum of charges = 0 we can define kind of a collection, that has the property of a single root of unity. The process shall be illustrated by one of the gluons, but can easily be adapted to the type of neutrinos.

Let $\boldsymbol{Z} \coloneqq \{\boldsymbol{z}_{\mathrm{i}}\ of\ type\ \boldsymbol{z} \coloneqq \frac{1}{\sqrt{6}}(\boldsymbol{e}_4 + \boldsymbol{e}_8 - 2\boldsymbol{e}_{12})$ with $\boldsymbol{z}_{\mathrm{i}}$ all different in some region of space-time} and let $\boldsymbol{Z}_{\mathrm{n}}$ be a finite subset of n elements of $\boldsymbol{Z}$ where $n = m^2$ with $m \in \mathbb{N}$. Then $\boldsymbol{E} \coloneqq \{< \boldsymbol{e}_0, \boldsymbol{z} >$[6]$: \boldsymbol{z} \in \boldsymbol{Z}_{\mathrm{n}}\}$ defines a finite set of planes all parallel to each other if we collapse other dimensions to one line, but crossing different points on that line (according to the selected $\boldsymbol{z}_{\mathrm{i}}$), each one corresponding to a root of unity of neutrino type. Now, set: $\boldsymbol{z}' \coloneqq \frac{1}{\sqrt{6}}(m\boldsymbol{e}_4 + m\boldsymbol{e}_8 - 2m\boldsymbol{e}_{12})$ and $\boldsymbol{z} \coloneqq \sum_{i=1}^{n} \boldsymbol{z}'(location)$ Then $\frac{1}{n}\sum_{i=1}^{n}(\frac{m}{\sqrt{6}} + \frac{m}{\sqrt{6}} - \frac{2m}{\sqrt{6}}) = 0$ and $\frac{1}{n^2}\sum_{i=1}^{n}((\frac{m}{\sqrt{6}})^2 + \left(\frac{m}{\sqrt{6}}\right)^2 + \left(\frac{2m}{\sqrt{6}}\right)^2) = \frac{1}{n^2} * n * m^2 = 1$. Since the square n may be arbitrary big, we possibly have an example of very large black holes without charge – if mass is not zero, although charge is.

So, for type neutrino the constructed summed root of unity with n summands could be a relic of the phase shortly after Big Bang. Neutrino-type black holes should not emerge in an expanding universe (density much too small), but they could be produced in the very early universe, before CMB. So, they can be candidates for seeds of the supermassive black holes, that are central to every galaxy in our universe. If one finds early galaxies very shortly after CMB, this could be an indication for this hypothesis.

Proton and neutron

We have already seen, that $\boldsymbol{z} \coloneqq \frac{2}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 - \frac{1}{3}\boldsymbol{e}_3$ is the only solution for a summed root of unity using 3 base elements with summed charge equal to 1. Define:

$\boldsymbol{p}^{+} \coloneqq cos\varphi\boldsymbol{e}_0 + sin\varphi\boldsymbol{z}$ to be the proton.

For 6 base elements (such that colours sum up to white) we have a solution:

$\boldsymbol{n} \coloneqq cos\varphi\boldsymbol{e}_0 + sin\varphi\boldsymbol{z}$ with $\boldsymbol{z} \coloneqq \frac{1}{6}(2\boldsymbol{e}_1 + 2\boldsymbol{e}_2 - 1\boldsymbol{e}_3 - 3\boldsymbol{e}_4 + 3\boldsymbol{e}_8 - 3\boldsymbol{e}_{12}) = \frac{1}{2}(\frac{2}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 - \frac{1}{3}\boldsymbol{e}_3 - \boldsymbol{e}_4 + \boldsymbol{e}_8 - \boldsymbol{e}_{12})$. This way, the neutron may be viewed as containing a proton, an electron and an anti-e-neutrino. This indicates, that the weak force may have to do with the addition in $\mathbb{S}$. And just like with neutrino-type black holes we can construct kind of a neutron-type black hole. Let m, n $\in \mathbb{N}$, $n = m^2$. Set: $\boldsymbol{z}' \coloneqq \frac{1}{6}(2m\boldsymbol{e}_1 + 2m\boldsymbol{e}_2 - 1m\boldsymbol{e}_3 - 3m\boldsymbol{e}_4 + 3m\boldsymbol{e}_8 - 3m\boldsymbol{e}_{12})$ and $\boldsymbol{z} \coloneqq \sum_{i=1}^{n} \boldsymbol{z}'(location)$ Then $\frac{1}{n}\sum_{i=1}^{n}(\frac{2}{6}m + \frac{2}{6}m - \frac{1}{6}m - \frac{3}{6}m + \frac{3}{6}m - \frac{3}{6}m) = 0$ and $\frac{1}{n^2}\sum_{i=1}^{n}((\frac{2}{6}m)^2 + \left(\frac{2}{6}m\right)^2 + \left(\frac{1}{6}m\right)^2 + \left(\frac{3}{6}m\right)^2 + \left(\frac{3}{6}m\right)^2 + \left(\frac{3}{6}m\right)^2) = \frac{1}{n^2} * n * m^2 = 1$. Since the square n may be arbitrary big, we possibly have an example of very large black holes without charge. May be these objects can be between neutron stars and ordinary black holes, depending on the alignment (location parameter) of the roots of unity.

… work to be done …

6 $< \boldsymbol{e}_0, \boldsymbol{z} >$ defines the plane spanned by (time) $\boldsymbol{e}_0$ and summed root of unity $\boldsymbol{z}$

Atomic nuclei

This section (greyed) doubts that there are any quarks at all and is highly speculative.

Let us come to a main feature of the concept of roots of unity: It is *cascading*[7]. If for instance $\boldsymbol{p}^{+} \coloneqq cos\varphi\boldsymbol{e}_0 + sin\varphi\boldsymbol{z}$ with $\boldsymbol{z} \coloneqq \frac{2}{3}\boldsymbol{e}_1 + \frac{2}{3}\boldsymbol{e}_2 - \frac{1}{3}\boldsymbol{e}_3$, then 3 elements of a base of $\mathbb{O}$ are involved. But if $(\boldsymbol{p}^{+})^{2^n} = -1$ then $\boldsymbol{a}_{\mathrm{p}} \coloneqq (\boldsymbol{p}^{+})^{2^{n-1}}$ may be viewed again as a base element, which can be enhanced to a basic triple creating the whole algebra of $\mathbb{O}$. Remember, that all the potencies of $\boldsymbol{p}^{+}$ are located in a plane, spanned by $\boldsymbol{e}_0$ and $\boldsymbol{z}$. My expectation is, that using the concept of cascading, we can act as if we would have a "virtual base" of $\mathbb{O}$ of dimension say n and may construct summed roots of unity satisfying the 2 equations: $\frac{1}{n}\sum_{i=1}^{n} z_{\mathrm{i}} \in \{0, \pm 1\}$ and $\frac{1}{n^2}\sum_{i=1}^{n} z_{\mathrm{i}}^{\mathbf{2}} = 1$. This concept shall not be proved here, but illustrated on base of some examples. Note that for higher values of n, objects like the proton will appear in different "clothes" of z_{i}. More than that: An object like the neutron, not stable on its own, will be stable when part of a pair (neutron, proton). A sum corresponding to a virtual base will be written using ⊕ for summation instead of +, i.e. $\boldsymbol{z}_{\mathrm{n}} \oplus \boldsymbol{z}_{\mathrm{p}}$ for neutron and proton.

The basic idea in construction of nuclei is to build some kind of virtual roots of unity on base of pairs of proton and neutron and occasionally additional neutrons. The components z_i of summed roots of unity for proton and neutron may be adapted, as long as they satisfy the summation rules for charge (±1 or 0) and the overall sum of squares, weighted by the quotient of the square of their number, leads to 1.

Example:

^2_1H : The nucleus of Deuterium corresponds to a pair of neutron and proton. If we define $\boldsymbol{z}_{\mathrm{n}} \coloneqq \frac{1}{6}(4\boldsymbol{e}_1 - 2\boldsymbol{e}_2 - 2\boldsymbol{e}_3)$ and $\boldsymbol{z}_{\mathrm{p}} \coloneqq \frac{1}{6}(2\boldsymbol{e}_1 + 2\boldsymbol{e}_2 + 2\boldsymbol{e}_3)$, we get for the 2 equations of the pair $\boldsymbol{z}_{\mathrm{n}} \oplus \boldsymbol{z}_{\mathrm{p}}$: $\frac{1}{6}(4 - 2 - 2 + 2 + 2 + 2) = 1 \in \{0, \pm 1\}$ and $\frac{1}{36}(16 + 4 + 4 + 4 + 4 + 4) = 1$. So, the neutron in a pair together with a proton will be stable.

^4_2He : Modify the summed root for the proton: $\boldsymbol{z} \coloneqq \frac{1}{3}(2\boldsymbol{e}_1 + 2\boldsymbol{e}_2 - 1\boldsymbol{e}_3)$ to the "pair-variant" $\boldsymbol{z}_{\mathrm{p}} \coloneqq \frac{1}{6}(4\boldsymbol{e}_1 + 1\boldsymbol{e}_2 + 1\boldsymbol{e}_3)$ and the one for the neutron: $\boldsymbol{z} \coloneqq \frac{1}{3}(2\boldsymbol{e}_1 - 1\boldsymbol{e}_2 - 1\boldsymbol{e}_3)$ to the "pair-variant": $\boldsymbol{z}_{\mathrm{n}} \coloneqq \frac{1}{6}(6\boldsymbol{e}_1 - 3\boldsymbol{e}_2 - 3\boldsymbol{e}_3)$. Then we get for the formal sum (coefficients of the $\boldsymbol{e}_{\mathrm{j}}$ remain separated!): $\frac{1}{2}(\boldsymbol{z}_{\mathrm{n}} \oplus \boldsymbol{z}_{\mathrm{p}} \oplus \boldsymbol{z}_{\mathrm{n}} \oplus \boldsymbol{z}_{\mathrm{p}})$ the weighted sum of coefficients: $\frac{1}{2} * \frac{1}{6}(6 - 3 - 3 + 4 + 1 + 1 + 6 - 3 - 3 + 4 + 1 + 1) = 1 \in \{0, \pm 1\}$ and for the weighted sum of squares: $\frac{1}{4} * \frac{1}{36}(36 + 9 + 9 + 16 + 1 + 1 + 36 + 9 + 9 + 16 + 1 + 1) = 1$.

^6_3Li : This time, we have to modify to triple variants. We choose: $\boldsymbol{z}_{\mathrm{p}} \coloneqq \frac{1}{6}(6\boldsymbol{e}_1 + 3\boldsymbol{e}_2 - 3\boldsymbol{e}_3)$ and $\boldsymbol{z}_{\mathrm{n}}$ like before. Then we get for the formal sum: $\frac{1}{3}(\boldsymbol{z}_{\mathrm{n}} \oplus \boldsymbol{z}_{\mathrm{p}} \oplus \boldsymbol{z}_{\mathrm{n}} \oplus \boldsymbol{z}_{\mathrm{p}} \oplus \boldsymbol{z}_{\mathrm{n}} \oplus \boldsymbol{z}_{\mathrm{p}})$ the weighted sum of coefficients: $\frac{1}{3} * \frac{1}{6}(6 - 3 - 3 + 6 + 3 - 3 + 6 - 3 - 3 + 6 + 3 -$

[7] May be a better approach would be via roots of unity separated by their locations in spacetime, similar to the neutrino-type black holes

$3+6-3-3+6+3-3)=1 \in \{0,\pm1\}$ and for weighted sum of squares: $\frac{1}{9}*\frac{1}{36}(6*(36+9+9))=1$.

An example for an additional neutron to a sum of pairs of proton and neutron is:

${}^{7}_{3}Li$: We choose: $\boldsymbol{z}_{\mathrm{p}} := \frac{1}{7}(6\boldsymbol{e}_1+2\boldsymbol{e}_2-1\boldsymbol{e}_3)$ and $\boldsymbol{z}_{\mathrm{n}} := \frac{1}{7}(7\boldsymbol{e}_1-4\boldsymbol{e}_2-3\boldsymbol{e}_3)$ and for the additional neutron $\boldsymbol{z}_{\mathrm{a}} := \frac{1}{7}(8\boldsymbol{e}_1-4\boldsymbol{e}_2-4\boldsymbol{e}_3)$. Then we get for the formal sum: $\frac{1}{3}(\boldsymbol{z}_{\mathrm{a}}\oplus\boldsymbol{z}_{\mathrm{n}}\oplus\boldsymbol{z}_{\mathrm{p}}\oplus\boldsymbol{z}_{\mathrm{n}}\oplus\boldsymbol{z}_{\mathrm{p}}\oplus\boldsymbol{z}_{\mathrm{n}}\oplus\boldsymbol{z}_{\mathrm{p}})$ the weighted sum of coefficients: $\frac{1}{3}*\frac{1}{7}(8-4-4+7-4-3+6+2-1+7-4-3+6+2-1+7-4-3+6+2-1)=1 \in \{0,\pm1\}$ and for weighted sum of squares: $\frac{1}{9}*\frac{1}{49}(64+16+16+3*(49+16+9)+3*(36+4+1))=1$. This element is also stable. It is even more present on earth than the isotope ${}^{6}_{3}Li$. What could be the reason? If we define the “energy level” of an isotope to be the sum of squares of the coefficients on charges, we get for ${}^{6}_{3}Li$: $3*(36+9+9)=3*54$ and for ${}^{7}_{3}Li$: $3*(36+4+1)=3*41$. So, the energy level of ${}^{7}_{3}Li$ would be lower than that of ${}^{6}_{3}Li$.

Note, that by now it is not clear, how grouping of neutrons, pairs of neutron and proton and higher order nuclei may be accomplished due to cascading (what sort of combination will be allowed? Solutions for the mere conditions $\frac{1}{n}\sum_{i=1}^{n} z_{\mathrm{i}} \in \{0,\pm1\}$ and $\frac{1}{n^2}\sum_{i=1}^{n} z_{\mathrm{i}}^{\mathbf{2}}=1$ seem to get more numerous for increasing n). So, the examples above are really not more than some examples.

On mass and charge

For a root of unity charge is given by construction:

If the summed root of unity part $\boldsymbol{z}$ within the root of unity $\boldsymbol{a} = cos\varphi\boldsymbol{e}_0 + sin\varphi\boldsymbol{z}$ is defined such that its sum of coefficients is equal to ±1 then this is called its »charge value«. If the sum of coefficients is equal to 0 then the root of unity is called »neutral«.

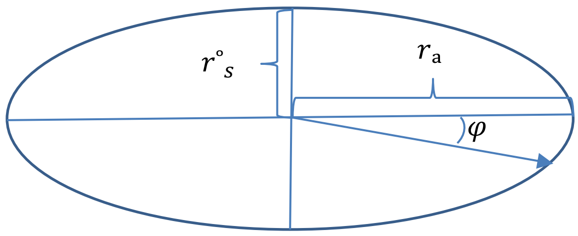
I assume, that the mass of a root of unity is given by our view on it with respect to dimensions integrated in the imaginary part of the root of unity. First note, that we can rewrite the equation of a root of unity to:

$$\boldsymbol{a} = cos\varphi\boldsymbol{e}_0 + sin\varphi\boldsymbol{z} = \pm\frac{\mathrm{r_a}}{\sqrt{1+\mathrm{r_a}^4}}\left(\mathrm{r_a}\boldsymbol{e}_0 + \frac{1}{\mathrm{r_a}}\boldsymbol{z}\right)$$

Let me illustrate this for the electron. Here the imaginary part is given through the extra dimension $\boldsymbol{e}_4$ within the octonions containing our 4-dimensional spacetime $\mathbb{H}$. I assume that the electron given by $\boldsymbol{e}^{-} := cos\varphi\boldsymbol{e}_0 - sin\varphi\boldsymbol{e}_4$ (2-dimensional!) induces a curvature within $\mathbb{H}$ through smooth continuation, that seems to show an ellipse encapsulating the root of unity, but an ellipse with extreme semi-axes. One of the semi-axes shall be given by the Compton-radius[8] $r_{\mathrm{a}} := \frac{\hbar}{m_{\mathrm{e}}c}$ with reduced Planck-quantum ħ, mass m_{e} of the electron and speed of light c, the other by $r^{\circ}{}_{S} := \frac{Gm_{\mathrm{e}}}{c^2}$ with gravitational constant G (this is the limit

[8] Compton wavelength is given by $2\pi r_a$, and $r_a*\alpha^{-1}$, with fine structure constant α, just yields the so-called Bohr radius

Schwarzschild-radius of a black hole with maximal rotation). Then: $r_a * r^\circ_s = l_p^2$ with l_p being the Planck length.



φ negative because of negative charge of the electron

The parametric representation of our ellipse may be written as:

$view\ on\ \boldsymbol{e}^- = r_a * cos\varphi \boldsymbol{e}_0 - r^\circ_s * sin\varphi \boldsymbol{e}_4$

For the electron we get: r_a =3,86159$_{267606883}$*10^{-13} [m] and r°_s=6,76455$_{124470864}$*10^{-58} [m] ($\alpha r_a / r^\circ_s$ is just the quotient of electrostatic and gravitational "force" for the electron). In units of l_p: r_a =2,38926$_{177766351}$*10^{22} [l_p] and r°_s=4,18539$_{320114983}$*10^{-23} [l_p]. So, the electron in our view is more of a line than an ellipse and its parametric representation in units of l_p may be written as:

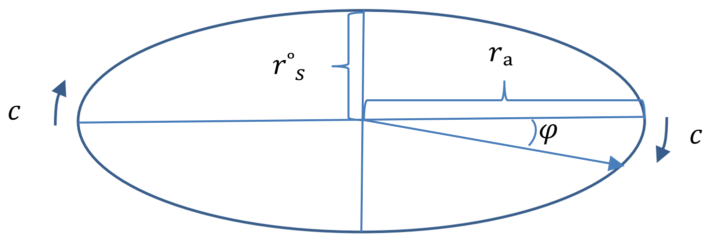
$view\ on\ \boldsymbol{e}^- = r_a * cos\varphi \boldsymbol{e}_0 - r_a^{-1} * sin\varphi \boldsymbol{e}_4$

In essence, masses of elementary particles would be given by our view on dimensions involved in the imaginary part of the associated root of unity. Note however, that when looking at the electron from the outside, roles of time and spacelike radius (including extra dimensions) change. So, from the outside the small semi-axis will be associated with time.

… work to be done …

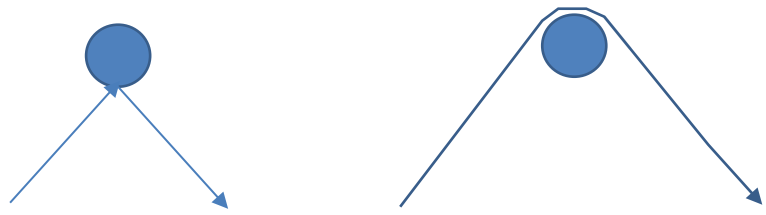
On Heisenberg's uncertainty principle

The content of the section above offers us a different approach to Heisenberg's uncertainty principle. Let us reflect the picture of our view on the electron:



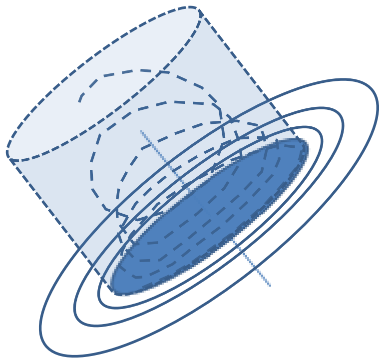
Then horizontal time axis inside the black hole corresponds to an axis in space from an outer perspective. Since we cannot look inside the black hole, its location is defined up to a precision of $\Delta s \geq r_a$ and impulse up to $\Delta p \geq m_e(r^\circ_s) * c$ (since the black hole is of maximum rotation and at the possible interaction points left and right on the horizontal axis tangential velocity is c (with opposite directions)). So, $\Delta s * \Delta p \geq \frac{\hbar}{m_e(r^\circ_s)c} * m_e(r^\circ_s) * c = \hbar$.

The value is higher by a factor of 2 than in standard theory. May be, that results from a different view in this model on the phenomenon of physical scattering of a photon by an electron. Scattering is not due to some kind of ping pong but due to curvature in the environment of the electron, as the following picture illustrates:



scattering in standard theory scattering due to curvature of black hole in this theory

Note, that embedding in spacetime yields for lines of equal curvature in a covering with non-intersecting curves of equal curvature spirals around the vertical axis on the “middle” of the black hole, outside the plane of the black hole:



Lines of equal curvature within the torus on the surface of the black hole have to be spiral because of the limit of speed of light at the boundary of the black hole with maximal rotation.

Planck view

In Planck view our ellipse transforms into a circle, and we get $r_a = r^\circ_s = l_p = \sqrt{\frac{\hbar G}{c^3}}$ and mass becomes the Planck mass $m_p = \sqrt{\frac{\hbar c}{G}}$. So, $\Delta s * \Delta p \geq l_p * m_p * c = \hbar$, i.e. the uncertainty principle remains true even in Planck view.

Objects

It seems, that the uncertainty principle results from our deficit to define what is an object, and there is nothing mysterious with it. Objects, whether microscopic or macroscopic, are only defined together with an uncertainty (see page 8 of the agenda of the project described in [(PDF) Our universe is just curved spacetime Our universe is just curved spacetime (researchgate.net)](); there, difficulties in defining some macroscopic objects have been described).

Objects are always objects in relation to our view of them. Every physical system thus is only defined with respect to the automorphisms that act on it and its symmetries/rules. So, uncertainty with respect to a physical system of elementary particles is given by its automorphism group.

On entanglement

One of the most curious phenomena of quantum theory is entanglement. Einstein once called this »spukhafte Fernwirkung« (spooky action at a distance), since states of combined Quantum objects stay related even when separated in space. Meaning, that measurement of certain states of one object, although random, fixes the ones of the other in similar measurement. How can the model presented here explain the situation?

The model states, that interaction of elementary particles includes interaction with spacetime. Remember, that with

$\boldsymbol{\partial x} = \sum_{i=0}^{3} \partial x_i \boldsymbol{e}_i \in \mathbb{H}, \partial x_i \in \mathbb{R}, \boldsymbol{e}_i$ building a basis of $\mathbb{H}, 0 \leq i \leq 3$, and $\partial(ct) \coloneqq \partial x_0$, using the multiplication in $\mathbb{H}$, we got:

$$\boldsymbol{\partial x^2} = (\partial(ct))^2 - ({\partial x_1}^2 + {\partial x_2}^2 + {\partial x_3}^2))\boldsymbol{e}_0 + 2\partial(ct)(\partial x_1 \boldsymbol{e}_1 + \partial x_2 \boldsymbol{e}_2 + \partial x_3 \boldsymbol{e}_3)$$

The time coordinate ($\boldsymbol{e}_0$) just describes the ordinary Minkowski metric, the space part kind of an expansion of space. If we look at the interaction of photons and spacetime, we get:

$\boldsymbol{e}^+ \boldsymbol{\partial x^2} \boldsymbol{e}^- = (cos\varphi \boldsymbol{e}_0 + sin\varphi \boldsymbol{e}_4) l^2 [(\partial(ct))^2 - \quad ({\partial x_1}^2 + {\partial x_2}^2 + {\partial x_3}^2))\boldsymbol{e}_0 + 2\partial(ct)(\partial x_1 \boldsymbol{e}_1 + \partial x_2 \boldsymbol{e}_2 + \partial x_3 \boldsymbol{e}_3)(cos\varphi \boldsymbol{e}_0 - sin\varphi \boldsymbol{e}_4)]$ with l^2 being associated with the wavelength. Now, obviously this is invariant under translations (part of G_{15}, added by Poincaré to Lorentz group). So, entangled positron and electron may be arbitrarily separated within spacetime.

… work to be done …

Outlook

Consequences

A lot of consequences result. Some of them:

- The picture of big bang, drawn by the theory presented here, could imply, that the theory of inflation in the very early phase of our universe is back, and may be specified more precisely on base of higher dimensions, existing for a very short time.
- The theory states a difference in generation III compared to generations I and II, as it contains zero divisors. Among the pairs of zero divisors, listed in [7] , the following products of nonzero elements of generation III can be found:

$$(\boldsymbol{e}_1 + \boldsymbol{e}_{12})(\boldsymbol{e}_2 + \boldsymbol{e}_{15}) = \boldsymbol{e}_3 - \boldsymbol{e}_{14} + \boldsymbol{e}_{14} - \boldsymbol{e}_3 = 0$$
$$(\boldsymbol{e}_1 + \boldsymbol{e}_{12})(\boldsymbol{e}_3 - \boldsymbol{e}_{14}) = 0$$
$$(\boldsymbol{e}_1 + \boldsymbol{e}_{13})(\boldsymbol{e}_2 - \boldsymbol{e}_{14}) = 0$$
$$(\boldsymbol{e}_1 + \boldsymbol{e}_{13})(\boldsymbol{e}_3 - \boldsymbol{e}_{15}) = 0$$
$$(\boldsymbol{e}_1 + \boldsymbol{e}_{14})(\boldsymbol{e}_2 + \boldsymbol{e}_{13}) = 0$$
$$(\boldsymbol{e}_1 + \boldsymbol{e}_{14})(\boldsymbol{e}_3 + \boldsymbol{e}_{12}) = 0$$
$$(\boldsymbol{e}_1 + \boldsymbol{e}_{15})(\boldsymbol{e}_3 + \boldsymbol{e}_{13}) = 0$$

$$(\boldsymbol{e}_1+\boldsymbol{e}_{15})(\boldsymbol{e}_2-\boldsymbol{e}_{12}) = 0$$
$$(\boldsymbol{e}_2+\boldsymbol{e}_{12})(\boldsymbol{e}_3+\boldsymbol{e}_{13}) = 0$$
$$(\boldsymbol{e}_2+\boldsymbol{e}_{13})(\boldsymbol{e}_3-\boldsymbol{e}_{12}) = 0$$
$$(\boldsymbol{e}_2+\boldsymbol{e}_{14})(\boldsymbol{e}_3-\boldsymbol{e}_{15}) = 0$$
$$(\boldsymbol{e}_2+\boldsymbol{e}_{15})(\boldsymbol{e}_3+\boldsymbol{e}_{14}) = 0$$

Note, that each part of pairs of zero divisors has an inverse. For example:

$(\boldsymbol{e}_1+\boldsymbol{e}_{12})(-\frac{1}{2}(\boldsymbol{e}_1+\boldsymbol{e}_{12}))=\boldsymbol{e}_0$.

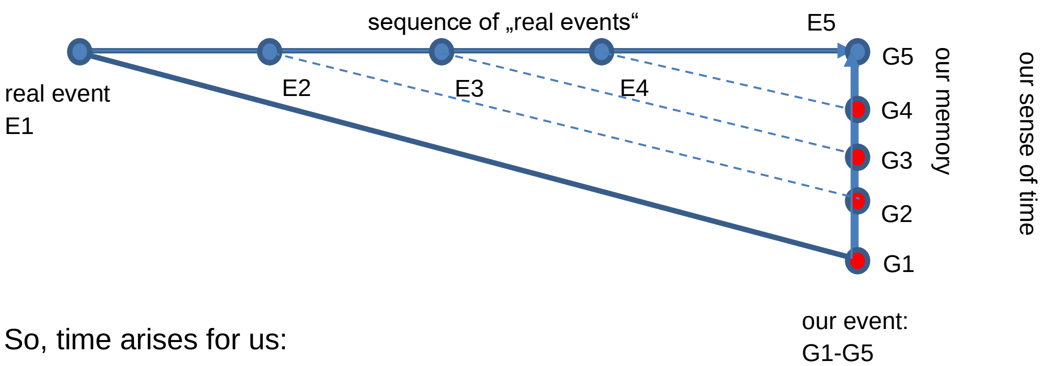
I expect, that conservation rules in quantum theory result from avoiding zero divisors in possible interactions. From a Darwinian standpoint of view the conservation rules are just part of our view, since we just cannot see interactions with zero divisors. It does not mean, that they do not take place.

… work to be done …

Expectations

I have some expectations on the theory presented. Some of them:

- I expect, that the rules of quantum theory are contained within the rules to avoid products of zero divisors in lifecycles and interactions of black holes of type elementary particle. This is because, “objects” created in the evolution of our universe, not following the rules of avoidance, would have disappeared. And what we see now is, what did survive.
- I expect, that the model could give an alternative explanation for the concept of superposition and decoherence. I already mentioned, that spin-½-objects, when interacting at point $\boldsymbol{x} \in \mathbb{H}$ of spacetime by “joining the reals” (time), do first at -1 then at 1. So, interaction takes place in the past and then in the future of time NOW. This could mean, that in an avalanche of time, our story in the past and in the future are rewritten completely (as shown in the picture on page 8). But, due to the finiteness of the speed of light, this takes time. Our interpretation results in a certain fuzziness of causality, since we expect the past to be determined. Why do we? The philosopher Husserl once created a model, for what we see as an event in time. The following drawing illustrates his idea (see [11]).



So, time arises for us:

1) because events will be stored in our memory, including information on predecessors and successors (sometimes not very solidly)
2) because stored events will be reflected in their original order (hopefully)
3) because „time“, more precisely: duration, will be associated to events according to the relevance of the events for the individual

4) because rules of causality will be associated (in a coordinated process)
5) because we derive forecasts for the future on base of the known events stored in our memory (Will the lion jump? Is it good, that my friend is walking before me?)
6) because real events will be (as G1-G5) organized by us as one (human) event according to event models stored in our brain (this point is not due to Husserl, rather it reflects a theory of the psychologist Radvansky, originally called event-horizon-model). In this model, an event is "closed", when for instance we change a room and closing the door behind us.

If we inspect smallest structures of our universe, we will wonder that causality seems to get lost. But on the layer of shapes of views, where fuzziness is just made (by us) as big as necessary to make the story fit, we will not be able to recognize non causality in the story that led to our view on the universe. More than that, the past will be fixed for us, because otherwise we could not have causality and therefore no forecast and no action planning. And that would be kind of horror for us. The fear of loss of control made us believe in gods like Zeus (Greeks) or that the pharaoh makes the sun shine again the next morning (ancient Egypt), and we will not get lost in darkness. We are children of our evolution.

Questions

A lot of questions remain. Some of them:

- Is this theory, that combines special relativity, derived gravity and inclusion of quantum theory on base of theory of black holes, consistent and does it fit with the huge number of observations?
- The subalgebra corresponding to generation III of the sedenions is not isomorphic to $\mathbb{O}$. What is the type of algebra, it is isomorphic to? See appendix A1 for some properties of this algebra.

Appendices

A Properties of the sedenions

In this section some essential properties of the sedenions will be derived. First, we have a look on its special subalgebra $\mathbb{O}_3$.

A1 Generation III: $\mathbb{O}_3$

Subalgebra $\mathbb{O}_3 \subset \mathbb{S}$ is not isomorphic to the algebra of octonions $\mathbb{O}$. It is an 8-dimensional subalgebra, which even contains zero divisors. In [8] the characterizing loop $\pm\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_{12}, \boldsymbol{e}_{13}, \boldsymbol{e}_{14}, \boldsymbol{e}_{15}\}$ of $\mathbb{O}_3 \subset \mathbb{S} \subset \mathbb{T}$ of order 16 is named *quasi-octonion*. Here $\mathbb{T}$ denotes the 32-dimensional Cayley-Dickson-algebra of trigintaduonions, the research object in [8].

When replacing indices 12-15 by 4-7, we can describe the multiplication by means of the following multiplication table:

*	$\boldsymbol{e}_0$	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_3$	$\boldsymbol{e}_4$	$\boldsymbol{e}_5$	$\boldsymbol{e}_6$	$\boldsymbol{e}_7$
$\boldsymbol{e}_0$	$\boldsymbol{e}_0$	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_3$	$\boldsymbol{e}_4$	$\boldsymbol{e}_5$	$\boldsymbol{e}_6$	$\boldsymbol{e}_7$
$\boldsymbol{e}_1$	$\boldsymbol{e}_1$	$-\boldsymbol{e}_0$	$\boldsymbol{e}_3$	$-\boldsymbol{e}_2$	$-\boldsymbol{e}_5$	$\boldsymbol{e}_4$	$\boldsymbol{e}_7$	$-\boldsymbol{e}_6$
$\boldsymbol{e}_2$	$\boldsymbol{e}_2$	$-\boldsymbol{e}_3$	$-\boldsymbol{e}_0$	$\boldsymbol{e}_1$	$-\boldsymbol{e}_6$	$-\boldsymbol{e}_7$	$\boldsymbol{e}_4$	$\boldsymbol{e}_5$
$\boldsymbol{e}_3$	$\boldsymbol{e}_3$	$\boldsymbol{e}_2$	$-\boldsymbol{e}_1$	$-\boldsymbol{e}_0$	$-\boldsymbol{e}_7$	$\boldsymbol{e}_6$	$-\boldsymbol{e}_5$	$\boldsymbol{e}_4$
$\boldsymbol{e}_4$	$\boldsymbol{e}_4$	$\boldsymbol{e}_5$	$\boldsymbol{e}_6$	$\boldsymbol{e}_7$	$-\boldsymbol{e}_0$	$-\boldsymbol{e}_1$	$-\boldsymbol{e}_2$	$-\boldsymbol{e}_3$
$\boldsymbol{e}_5$	$\boldsymbol{e}_5$	$-\boldsymbol{e}_4$	$\boldsymbol{e}_7$	$-\boldsymbol{e}_6$	$\boldsymbol{e}_1$	$-\boldsymbol{e}_0$	$\boldsymbol{e}_3$	$-\boldsymbol{e}_2$
$\boldsymbol{e}_6$	$\boldsymbol{e}_6$	$-\boldsymbol{e}_7$	$-\boldsymbol{e}_4$	$\boldsymbol{e}_5$	$\boldsymbol{e}_2$	$-\boldsymbol{e}_3$	$-\boldsymbol{e}_0$	$\boldsymbol{e}_1$
$\boldsymbol{e}_7$	$\boldsymbol{e}_7$	$\boldsymbol{e}_6$	$-\boldsymbol{e}_5$	$-\boldsymbol{e}_4$	$\boldsymbol{e}_3$	$\boldsymbol{e}_2$	$-\boldsymbol{e}_1$	$-\boldsymbol{e}_0$

$\mathbb{O}_3$

So, for 0<i<j<8 we have: $\boldsymbol{e}_i^2 = -\boldsymbol{e}_0$ and $\boldsymbol{e}_i\boldsymbol{e}_j = -\boldsymbol{e}_j\boldsymbol{e}_i$.

When do we have $\boldsymbol{a}(\boldsymbol{a}\boldsymbol{x}) = \boldsymbol{a}^2\boldsymbol{x}$ and $(\boldsymbol{x}\boldsymbol{a})\boldsymbol{a} = \boldsymbol{x}\boldsymbol{a}^2$, provided $\boldsymbol{x} \in \mathbb{H} \subset \mathbb{O}_3$?

Let $\boldsymbol{a} := \sum_{i=0}^{7} a_i\boldsymbol{e}_i \in \mathbb{O}_3$ and $\boldsymbol{x} := \sum_{i=0}^{3} x_i\boldsymbol{e}_i \in \mathbb{H} \subset \mathbb{O}_3$ and $r^2 := \sum_{i=1}^{7} a_i^2$. Then:

$$\boldsymbol{a}(\boldsymbol{a}\boldsymbol{x}) = \left(\sum_{i=0}^{7} a_i\boldsymbol{e}_i\right)\left(\left(\sum_{i=0}^{7} a_i\boldsymbol{e}_i\right)\left(\sum_{i=0}^{3} x_i\boldsymbol{e}_i\right)\right) =$$

$$((a_0^2 - r^2)x_0 - 2a_0a_1x_1 - 2a_0a_2x_2 - 2a_0a_3x_3)\boldsymbol{e}_0 +$$
$$(2a_0a_1x_0 + (a_0^2 - r^2)x_1 - 2a_0a_3x_2 + 2a_0a_2x_3)\boldsymbol{e}_1 +$$
$$(2a_0a_2x_0 + 2a_0a_3x_1 + (a_0^2 - r^2)x_2 - 2a_0a_1x_3)\boldsymbol{e}_2 +$$
$$(2a_0a_3x_0 - 2a_0a_2x_1 + 2a_0a_1x_2 + (a_0^2 - r^2)x_3)\boldsymbol{e}_3 +$$
$$(2a_0a_4x_0 + (-2a_0a_5 - 2a_3a_6 + 2a_2a_7)x_1 + (-2a_0a_6 - 2a_1a_7 + 2a_3a_5)x_2 + (-2a_0a_7 + 2a_1a_6 - 2a_2a_5)x_3)\boldsymbol{e}_4 +$$
$$(2a_0a_5x_0 + (2a_0a_4 - 2a_2a_6 - 2a_3a_7)x_1 + (-2a_0a_7 + 2a_1a_6 - 2a_3a_4)x_2 + (2a_0a_6 + 2a_1a_7 + 2a_2a_4)x_3)\boldsymbol{e}_5 +$$
$$(2a_0a_6x_0 + (2a_0a_7 + 2a_2a_5 + 2a_3a_4)x_1 + (2a_0a_4 - 2a_1a_5 - 2a_3a_7)x_2 + (-2a_0a_5 - 2a_1a_4 + 2a_2a_7)x_3)\boldsymbol{e}_6 +$$
$$(2a_0a_7x_0 + (-2a_0a_6 - 2a_2a_4 + 2a_3a_5)x_1 + (2a_0a_5 + 2a_1a_4 + 2a_3a_6)x_2 + (2a_0a_4 - 2a_1a_5 - 2a_2a_6)x_3)\boldsymbol{e}_7$$

$\boldsymbol{a^2x} = \left(\left(\sum_{i=0}^{7} a_i\boldsymbol{e}_i\right)\left(\sum_{i=0}^{7} a_i\boldsymbol{e}_i\right)\right)\left(\sum_{i=0}^{3} x_i\boldsymbol{e}_i\right) =$
$(x_0({a_0}^2 - r^2) + 2a_0(-a_1x_1 - a_2x_2 - a_3x_3))\boldsymbol{e}_0 +$
$(x_1({a_0}^2 - r^2) + 2a_0a_1x_0 - 2a_0a_3x_2 + 2a_0a_2x_3)\boldsymbol{e}_1 +$
$(x_2({a_0}^2 - r^2) + 2a_0a_2x_0 + 2a_0a_3x_1 - 2a_0a_1x_3)\boldsymbol{e}_2 +$
$(x_3({a_0}^2 - r^2) + 2a_0a_3x_0 - 2a_0a_2x_1 + 2a_0a_1x_2)\boldsymbol{e}_3 +$
$(2a_0a_4x_0 - 2a_0a_5x_1 - 2a_0a_6x_2 - 2a_0a_7x_3)\boldsymbol{e}_4 +$
$(2a_0a_5x_0 + 2a_0a_4x_1 - 2a_0a_7x_2 + 2a_0a_6x_3)\boldsymbol{e}_5 +$
$(2a_0a_6x_0 + 2a_0a_7x_1 + 2a_0a_4x_2 - 2a_0a_5x_3)\boldsymbol{e}_6 +$
$(2a_0a_7x_0 - 2a_0a_6x_1 + 2a_0a_5x_2 + 2a_0a_4x_3)\boldsymbol{e}_7$

So, $\boldsymbol{a}(\boldsymbol{ax}) = \boldsymbol{a^2x}$ if and only if the following equations hold:

	1	2	3
1	$a_2a_7 = a_3a_6$	$a_1a_7 = a_3a_5$	$a_1a_6 = a_2a_5$
2	$a_2a_6 = -a_3a_7$	$a_1a_7 = -a_2a_4$	$a_1a_6 = a_3a_4$
3	$a_2a_5 = -a_3a_4$	$a_1a_5 = -a_3a_7$	$a_1a_4 = a_2a_7$
4	$a_2a_4 = a_3a_5$	$a_1a_4 = -a_3a_6$	$a_1a_5 = -a_2a_6$

case 1: ${a_1}^2 + {a_2}^2 + {a_3}^2 \neq 0$
$\succ ({a_1}^2 + {a_2}^2 + 2{a_3}^2)a_7 = a_1a_3a_5 + a_2a_3a_6 - a_3a_2a_6 - a_3a_1a_5 = 0$ (using (1,2),(1,1),(3,2),(2,1)).
So, $a_7 = 0$. The same way we get:
$({a_1}^2 + {a_2}^2 + 2{a_3}^2)a_5 = -a_1a_3a_7 - a_2a_3a_4 + a_3a_2a_4 + a_3a_1a_7 = 0$ (using (3,2),(3,1),(4,1),(1,2)).
So, $a_5 = 0$.
$\succ$ From (3,1): $a_3a_4 = 0$, (4,1): $a_2a_4 = 0$, (3,3): $a_1a_4 = 0 \succ ({a_1}^2 + {a_2}^2 + {a_3}^2)a_4 = 0$,
hence: $a_4 = 0$, and the same way we get $a_6 = 0$. So, $\boldsymbol{a} \in \mathbb{H}$.

case 2: ${a_1}^2 + {a_2}^2 + {a_3}^2 = 0$
Then all the 12 equations are fulfilled.

Note, that there is no condition for a_0.

This way, the alternative identity $\boldsymbol{a}(\boldsymbol{ax}) = \boldsymbol{a^2x}$ holds for $\boldsymbol{x} \in \mathbb{H} \subset \mathbb{O}_3$ and ($\boldsymbol{a} \coloneqq \sum_{i=0}^{3} a_i\boldsymbol{e}_i \in \mathbb{H} \subset \mathbb{O}_3$ or $\boldsymbol{a} \coloneqq a_0\boldsymbol{e}_0 + \sum_{i=4}^{7} a_i\boldsymbol{e}_i \in \mathbb{O}_3$).

In a similar way we get for right alternative identity: $(\boldsymbol{xa})\boldsymbol{a} = \boldsymbol{xa^2}$ if and only if the following equations hold:

	1	2	3
1	$a_2a_7 = a_3a_6$	$a_1a_7 = a_3a_5$	$a_1a_6 = a_2a_5$
2	$a_3a_7 = -a_2a_6$	$a_1a_6 = a_3a_4$	$a_1a_7 = -a_2a_4$
3	$a_2a_5 = -a_3a_4$	$a_3a_7 = -a_1a_5$	$a_2a_7 = a_1a_4$
4	$a_3a_5 = a_2a_4$	$a_1a_4 = -a_3a_6$	$a_1a_5 = -a_2a_6$

which turns out to be exactly the same equations as for left alternative identity.
Remember, that $\boldsymbol{x} \in \mathbb{H} \subset \mathbb{O}_3$. Together:

Let $\boldsymbol{a} \in \mathbb{O}_3$ and $\boldsymbol{x} \in \mathbb{H} \subset \mathbb{O}_3$. Then $\boldsymbol{a}(\boldsymbol{ax}) = \boldsymbol{a^2x}$ and $(\boldsymbol{xa})\boldsymbol{a} = \boldsymbol{xa^2}$ if and only if $\boldsymbol{a} \in \mathbb{H}$ or $\boldsymbol{a} \in\ < \boldsymbol{e}_0, \boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{e}_6, \boldsymbol{e}_7 > \subset \mathbb{O}_3$., where $< \boldsymbol{e}_0, \boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{e}_6, \boldsymbol{e}_7 >$ is the vector space over the reals generated by $\boldsymbol{e}_0, \boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{e}_6, \boldsymbol{e}_7$.

Does $\mathbb{O}_3$ have features not valid for $\mathbb{O}_1$ and $\mathbb{O}_2$?

If we define $\boldsymbol{a} \coloneqq a_0\boldsymbol{e}_0 + \sum_{i=1}^{3} a_i\boldsymbol{e}_i \in \mathbb{H} \subset \mathbb{O}_3$ and $\boldsymbol{a}^{\bar{}} \coloneqq a_0\boldsymbol{e}_0 \pm \sum_{i=1}^{3} a_i\boldsymbol{e}_{4+i} \in\ < \boldsymbol{e}_0, \boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{e}_6, \boldsymbol{e}_7 > \subset \mathbb{O}_3$ and $\boldsymbol{x} \coloneqq \sum_{i=0}^{3} x_i\boldsymbol{e}_i \in \mathbb{H} \subset \mathbb{O}_3$, then we get the identity:

$(\boldsymbol{ax})\boldsymbol{a}^{-} = \boldsymbol{a}(\boldsymbol{x}\boldsymbol{a}^{-})$. It will be enough, to prove the identity for $\boldsymbol{a}' := \sum_{i=1}^{3} a_i \boldsymbol{e}_i$ and $\boldsymbol{a}'^{-} := \sum_{i=1}^{3} a_i \boldsymbol{e}_{4+i} \in < \boldsymbol{e}_5, \boldsymbol{e}_6, \boldsymbol{e}_7 > \subset \mathbb{O}_3$ and $\boldsymbol{x}' := \sum_{i=1}^{3} x_i \boldsymbol{e}_i$. We get:

$(\boldsymbol{a}'\boldsymbol{x}')\boldsymbol{a}'^{-} = ((-a_1{}^2 + a_2{}^2 + a_3{}^2)x_1 - 2a_1a_2x_2 - 2a_1a_3x_3)\boldsymbol{e}_5 + (-2a_1a_2x_1 + (a_1{}^2 - a_2{}^2 + a_3{}^2)x_2 - 2a_2a_3x_3)\boldsymbol{e}_6 + (-2a_1a_3x_1 - 2a_2a_3x_2 + (a_1{}^2 + a_2{}^2 - a_3{}^2)x_3)\boldsymbol{e}_7 = \boldsymbol{a}'(\boldsymbol{x}'\boldsymbol{a}'^{-})$

And this identity is *not* true for $\mathbb{O}_1$ and $\mathbb{O}_2$.

In $\mathbb{O}_1$ and $\mathbb{O}_2$ we have compatibility of the norm $\|\boldsymbol{a}\|^2 = \boldsymbol{a}\boldsymbol{a}^* \in \mathbb{R}_+$ with the multiplication, since both algebras are isomorphic to the normed algebra of octonions. When will this be true for $\mathbb{O}_3$? We have the following result:

For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{O}_3 \subset \mathbb{S}, \boldsymbol{a} := \sum_{i=0}^{7} a_i \boldsymbol{e}_i, \boldsymbol{b} := \sum_{i=0}^{7} b_i \boldsymbol{e}_i$ we have $\|\boldsymbol{ab}\|^2 = \|\boldsymbol{a}\|^2\|\boldsymbol{b}\|^2$, if and only if:

$(a_2b_1 - a_1b_2)(a_4b_7 - a_7b_4 + a_6b_5 - a_5b_6) + (a_1b_3 - a_3b_1)(a_4b_6 - a_6b_4 + a_5b_7 - a_7b_5) + (a_3b_2 - a_2b_3)(a_4b_5 - a_5b_4 + a_7b_6 - a_6b_7) = 0$ (A_3)

Proof:

A lengthy calculation yields:

$$\|\boldsymbol{ab}\|^2 = \|\boldsymbol{a}\|^2\|\boldsymbol{b}\|^2 + 4(-a_2a_4b_3b_5 + a_2a_5b_3b_4 + a_2a_6b_3b_7 - a_2a_7b_3b_6 + a_3a_4b_2b_5 - a_3a_5b_2b_4 - a_3a_6b_2b_7 + a_3a_7b_2b_6 - a_3a_4b_1b_6 - a_3a_5b_1b_7 + a_3a_6b_1b_4 + a_3a_7b_1b_5 + a_1a_4b_3b_6 + a_1a_5b_3b_7 - a_1a_6b_3b_4 - a_1a_7b_3b_5 - a_1a_4b_2b_7 + a_1a_5b_2b_6 - a_1a_6b_2b_5 + a_1a_7b_2b_4 + a_2a_4b_1b_7 - a_2a_5b_1b_6 + a_2a_6b_1b_5 - a_2a_7b_1b_4) = \|\boldsymbol{a}\|^2\|\boldsymbol{b}\|^2 + 4[(a_2b_1 - a_1b_2)(a_4b_7 - a_7b_4 + a_6b_5 - a_5b_6) + (a_1b_3 - a_3b_1)(a_4b_6 - a_6b_4 + a_5b_7 - a_7b_5) + (a_3b_2 - a_2b_3)(a_4b_5 - a_5b_4 + a_7b_6 - a_6b_7)]$$

Now, especially when $\boldsymbol{a}$ *or* $\boldsymbol{b} \in \mathbb{H} \subset \mathbb{O}_3$, or $\boldsymbol{a}$ *or* $\boldsymbol{b} \in < \boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{e}_6, \boldsymbol{e}_7 > \subset \mathbb{O}_3$, ($< \boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{e}_6, \boldsymbol{e}_7 >$ being the vector space complementary to $\mathbb{H}$ within $\mathbb{O}_3$, spanned by $\boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{e}_6, \boldsymbol{e}_7$), then identity ($A_3$) is fulfilled, hence: $\|\boldsymbol{ab}\|^2 = \|\boldsymbol{a}\|^2\|\boldsymbol{b}\|^2$. More generally:

If $(b_1, b_2, b_3) = \lambda(a_1, a_2, a_3)$ or $(a_1, a_2, a_3) = \lambda(b_1, b_2, b_3)$ or $(b_4, b_5, b_6, b_7) = \lambda(a_4, a_5, a_6, a_7)$ or $(a_4, a_5, a_6, a_7) = \lambda(b_4, b_5, b_6, b_7)$ with $\lambda \in \mathbb{R}$, then (A_3) is fulfilled, and we then have: $\|\boldsymbol{ab}\|^2 = \|\boldsymbol{a}\|^2\|\boldsymbol{b}\|^2$.

For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{O}_1 \cup \mathbb{O}_2 \subset \mathbb{S}$, the identity $\|\boldsymbol{ab}\|^2 = \|\boldsymbol{a}\|^2\|\boldsymbol{b}\|^2$ is valid, since $\mathbb{O}_1$ and $\mathbb{O}_2$ are isomorphic to the division algebra of octonions.

A2 Properties of $\mathbb{O}_1$, $\mathbb{O}_2$ and $\mathbb{O}_3$

We know, that beginning with the algebra $\mathbb{R}$ over $\mathbb{R}$ we can construct a chain of finite dimensional algebras over $\mathbb{R}$, using the Cayley-Dickson doubling mechanism: If $\mathbb{A}_n$ is an already constructed algebra with dimension 2^n over $\mathbb{R}$, we get the next algebra in the chain by building $\mathbb{A}_n \times \mathbb{A}_n$ together with the new multiplication for pairs of elements of $\mathbb{A}_n$: $(\boldsymbol{a}, \boldsymbol{b}) * (\boldsymbol{c}, \boldsymbol{d}) := (\boldsymbol{ac} - (\boldsymbol{d}^*)\boldsymbol{b}, \boldsymbol{da} + \boldsymbol{b}(\boldsymbol{c}^*))$, where $\boldsymbol{d}^*$ and $\boldsymbol{c}^*$ are the conjugate of $\boldsymbol{d}$ and $\mathbf{c}$, built from $\boldsymbol{d}$ and $\mathbf{c}$ by replacing their imaginary part with its negative. The new algebra shall be titled $\mathbb{A}_{n+1}$ since its vector space over $\mathbb{R}$ has dimension 2^{n+1}. $\mathbb{A}_n$ is then naturally embedded by the set $\{(\boldsymbol{a}, \boldsymbol{0}): \boldsymbol{a} \in \mathbb{A}_n\}$, since multiplication $(\boldsymbol{a}, \boldsymbol{0}) * (\boldsymbol{c}, \boldsymbol{0}) := (\boldsymbol{ac}, \boldsymbol{0})$ is just the multiplication of $\mathbb{A}_n$. This way we get a chain of Cayley-Dickson-algebras $\mathbb{A}_0 \subset \mathbb{A}_1 \subset \mathbb{A}_2 \subset \mathbb{A}_3 \subset \mathbb{A}_4 \subset \mathbb{A}_5 ... \subset \mathbb{A}_n \subset \mathbb{A}_{n+1}$ …. Here $\mathbb{A}_0 = \mathbb{R}$, $\mathbb{A}_1 = \mathbb{C}$, $\mathbb{A}_2 = \mathbb{H}$ (quaternions), $\mathbb{A}_3 = \mathbb{O}$ (octonions), $\mathbb{A}_4 = \mathbb{S}$ (sedenions), $\mathbb{A}_5 = \mathbb{T}$ (trigintaduonions, see [8]) in an arbitrary long chain. We have identified $\mathbb{A}_2 = \mathbb{H}$ with spacetime. The naturally embedded subalgebra $\mathbb{A}_0 = \mathbb{R}$ herein is the only ordered algebra in the chain. It is identified with time. Human kind may be glad to see, that the following is true (since stagnation in views will be avoided):

If for $\boldsymbol{a} \in \mathbb{A}_\mathrm{n}$ (2^n-dimensional Cayley-Dickson-algebra in the chain) with n>2 and $\boldsymbol{x} \in \mathbb{A}_2 = \mathbb{H}$, $\boldsymbol{a}, \boldsymbol{x} \neq \boldsymbol{0}$, then $\boldsymbol{ax} \neq \boldsymbol{0}$ and $\boldsymbol{xa} \neq \boldsymbol{0}$.

Proof:

n=3: Then we have $\mathbb{H}$ embedded in $\mathbb{O}$ and $\mathbb{O}$ is a division algebra.

n→n+1: Let $(\boldsymbol{a}, \boldsymbol{b})$, $(\boldsymbol{x}, \boldsymbol{0}) \in \mathbb{A}_{\mathrm{n}+1}$, with $\boldsymbol{x} \in \mathbb{A}^2 = \mathbb{H} \subset \mathbb{A}_\mathrm{n}$, $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{A}_\mathrm{n}$ and $(\boldsymbol{a}, \boldsymbol{b}), (\boldsymbol{x}, \boldsymbol{0}) \neq (\boldsymbol{0}, \boldsymbol{0})$. Then $(\boldsymbol{a}, \boldsymbol{b}) * (\boldsymbol{x}, \boldsymbol{0}) = (\boldsymbol{ax}, \boldsymbol{bx}^*)$. If $\boldsymbol{a} \neq \boldsymbol{0}$ then $\boldsymbol{ax} \neq \boldsymbol{0}$. Otherwise $\boldsymbol{b} \neq \boldsymbol{0} \succ \boldsymbol{bx} \neq \boldsymbol{0} \succ \boldsymbol{bx}^* \neq \boldsymbol{0}$.

The same way, one can show, that for $\boldsymbol{a} \in \mathbb{A}_\mathrm{n}$ (2^n-dimensional Cayley-Dickson-algebra in the chain) with n>2 and $\boldsymbol{x} \in \mathbb{A}_2 = \mathbb{H}$, $\boldsymbol{a}, \boldsymbol{x} \neq \boldsymbol{0}$, then $\boldsymbol{xa} \neq \boldsymbol{0}$.

Especially:

If for $\boldsymbol{a} \in \mathbb{S}$ (16-dimensional algebra of *sedenion*s) and $\boldsymbol{x} \in \mathbb{H} \subset \mathbb{S}$, $\boldsymbol{a}, \boldsymbol{x} \neq \boldsymbol{0}$, then $\boldsymbol{ax} \neq \boldsymbol{0}$.

On the other hand, for $\boldsymbol{a} \in \mathrm{H}$ (subalgebra of $\mathbb{S}$ generated by $\{\boldsymbol{e}_0, \boldsymbol{e}_4, \boldsymbol{e}_8, \boldsymbol{e}_{12}\}$ (isomorphic to $\mathbb{H}$)) and $\boldsymbol{y} \in \mathbb{S}$, we get $\boldsymbol{a}(\boldsymbol{ay}) = (\boldsymbol{aa})\boldsymbol{y}$, and therefore: If $\boldsymbol{a}, \boldsymbol{y} \neq \boldsymbol{0}$, then $\boldsymbol{ay} \neq \boldsymbol{0}$.

Proof:

Let $\boldsymbol{a} \coloneqq a_4\boldsymbol{e}_4 + a_8\boldsymbol{e}_8 + a_{12}\boldsymbol{e}_{12}$, $\boldsymbol{a}' \coloneqq a_0\boldsymbol{e}_0 + \boldsymbol{a} \in \mathrm{H}$ and $\boldsymbol{y} \coloneqq \sum_{i=1}^{15} y_\mathrm{i}\boldsymbol{e}_\mathrm{i}$, $\boldsymbol{y}' \coloneqq y_0\boldsymbol{e}_0 + \boldsymbol{y} \in \mathbb{S}$.
Then:

$\boldsymbol{a}'(\boldsymbol{a}'\boldsymbol{y}') = (a_0\boldsymbol{e}_0 + \boldsymbol{a})((a_0\boldsymbol{e}_0 + \boldsymbol{a})(y_0\boldsymbol{e}_0 + \boldsymbol{y})) = (a_0\boldsymbol{e}_0 + \boldsymbol{a})(a_0y_0\boldsymbol{e}_0 + a_0\boldsymbol{y} + y_0\boldsymbol{a} + \boldsymbol{ay}) = a_0{}^2y_0\boldsymbol{e}_0 + a_0{}^2\boldsymbol{y} + 2a_0y_0\boldsymbol{a} + 2a_0\boldsymbol{ay} + y_0\boldsymbol{a}^2 + \boldsymbol{a}(\boldsymbol{ay})$

$(\boldsymbol{a}'\boldsymbol{a}')\boldsymbol{y}' = ((a_0\boldsymbol{e}_0 + \boldsymbol{a})(a_0\boldsymbol{e}_0 + \boldsymbol{a}))(y_0\boldsymbol{e}_0 + \boldsymbol{y}) = (a_0{}^2\boldsymbol{e}_0 + 2a_0\boldsymbol{a} + \boldsymbol{a}^2)(y_0\boldsymbol{e}_0 + \boldsymbol{y}) = a_0{}^2y_0\boldsymbol{e}_0 + a_0{}^2\boldsymbol{y} + 2a_0y_0\boldsymbol{a} + 2a_0\boldsymbol{ay} + y_0\boldsymbol{a}^2 + \boldsymbol{a}^2\boldsymbol{y}$

So it is sufficient, to proof $\boldsymbol{a}(\boldsymbol{ay}) = \boldsymbol{a}^2\boldsymbol{y}$ for the elements without $\boldsymbol{e}_0$-part.

$(a_4\boldsymbol{e}_4 + a_8\boldsymbol{e}_8 + a_{12}\boldsymbol{e}_{12})((a_4\boldsymbol{e}_4 + a_8\boldsymbol{e}_8 + a_{12}\boldsymbol{e}_{12})(\sum_{i=1}^{15} y_\mathrm{i}\boldsymbol{e}_\mathrm{i})) = (a_4\boldsymbol{e}_4 + a_8\boldsymbol{e}_8 + a_{12}\boldsymbol{e}_{12}) * ((-a_4x_4 - a_8x_8 - a_{12}x_{12})\boldsymbol{e}_0 + (a_4x_5 + a_8x_9 - a_{12}x_{13})\boldsymbol{e}_1 + (a_4x_6 + a_8x_{10} - a_{12}x_{14})\boldsymbol{e}_2 + (a_4x_7 + a_8x_{11} - a_{12}x_{15})\boldsymbol{e}_3 + (a_8x_{12} - a_{12}x_8)\boldsymbol{e}_4 + (-a_4x_1 + a_8x_{13} + a_{12}x_9)\boldsymbol{e}_5 + (-a_4x_2 + a_8x_{14} + a_{12}x_{10})\boldsymbol{e}_6 + (-a_4x_3 + a_8x_{15} + a_{12}x_{11})\boldsymbol{e}_7 + (-a_4x_{12} + a_{12}x_4)\boldsymbol{e}_8 + (-a_4x_{13} - a_8x_1 - a_{12}x_5)\boldsymbol{e}_9 + (-a_4x_{14} - a_8x_2 - a_{12}x_6)\boldsymbol{e}_{10} + (-a_4x_{15} - a_8x_3 - a_{12}x_7)\boldsymbol{e}_{11} + (a_4x_8 - a_8x_4)\boldsymbol{e}_{12} + (a_4x_9 - a_8x_5 + a_{12}x_1)\boldsymbol{e}_{13} + (a_4x_{10} - a_8x_6 + a_{12}x_2)\boldsymbol{e}_{14} + (a_4x_{11} - a_8x_7 + a_{12}x_3)\boldsymbol{e}_{15}) = \cdots = -(a_4{}^2 + a_8{}^2 + a_{12}{}^2)(\sum_{i=1}^{15} y_\mathrm{i}\boldsymbol{e}_\mathrm{i}) = (a_4\boldsymbol{e}_4 + a_8\boldsymbol{e}_8 + a_{12}\boldsymbol{e}_{12})^2(\sum_{i=1}^{15} y_\mathrm{i}\boldsymbol{e}_\mathrm{i})$.

What about $\boldsymbol{axa}$?[9] Is $\boldsymbol{axa} \neq \boldsymbol{0}$ whenever $\boldsymbol{a} \in \mathbb{S}, \boldsymbol{x} \in \mathbb{H} \subset \mathbb{S}$, $\boldsymbol{a}, \boldsymbol{x} \neq \boldsymbol{0}$? Or is this even true, when $\boldsymbol{a} \in \mathbb{A}_\mathrm{n}$ (2^n-dimensional Cayley-Dickson-algebra in the chain)? This is not clear. But we can show the following:

If for $\boldsymbol{a} \in \mathbb{O}_1 \cup \mathbb{O}_2 \cup \mathbb{O}_3 \subset \mathbb{S}$ and $\boldsymbol{x} \in \mathbb{H} \subset \mathbb{S}$, $\boldsymbol{a}, \boldsymbol{x} \neq \boldsymbol{0}$, then $\boldsymbol{axa} \neq \boldsymbol{0}$.

Proof:

For $\boldsymbol{a} \in \mathbb{O}_1 \cup \mathbb{O}_2$ this is clear, since both $\mathbb{O}_1$ and $\mathbb{O}_2$ are division algebras containing $\mathbb{H}$. Let $\boldsymbol{a} \in \mathbb{O}_3$. Assume that $\boldsymbol{axa} = \boldsymbol{0}$, Then, after some exhausting calculation with $\boldsymbol{x} \coloneqq$

9 Note, that we can write this without brackets, since every Cayley-Dickson-algebra $\mathbb{A}_\mathrm{n}$ satisfies the flexible law $(\boldsymbol{ab})\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{ba})\ \forall\, \boldsymbol{a}, \boldsymbol{b} \in \mathbb{A}_\mathrm{n}$ (see [9]).

$\sum_{i=0}^{3} x_i \boldsymbol{e}_i \in \mathbb{H} \subset \mathbb{O}_3$ and $\boldsymbol{a} := \sum_{i=0}^{7} a_i \boldsymbol{e}_i \in \mathbb{O}_3$ (see appendix A1 for the multiplication table of $\mathbb{O}_3$), we end up with the following 8 equations:

(1) $(-a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2)x_0 + 2a_0(a_1x_1 + a_2x_2 + a_3x_3) = 0$
(2) $(a_0^2 - a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2)x_1 + 2a_1(a_0x_0 - a_2x_2 - a_3x_3) = 0$
(3) $(a_0^2 + a_1^2 - a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2)x_2 + 2a_2(a_0x_0 - a_1x_1 - a_3x_3) = 0$
(4) $(a_0^2 + a_1^2 + a_2^2 - a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2)x_3 + 2a_3(a_0x_0 - a_1x_1 - a_2x_2) = 0$
(5) $a_0a_4x_0 + (-a_1a_4 - a_2a_7 + a_3a_6)x_1 + (a_1a_7 - a_2a_4 - a_3a_5)x_2 + (-a_1a_6 + a_2a_5 - a_3a_4)x_3 = 0$
(6) $a_0a_5x_0 + (-a_1a_5 + a_2a_6 + a_3a_7)x_1 + (-a_1a_6 - a_2a_5 + a_3a_4)x_2 + (-a_1a_7 - a_2a_4 - a_3a_5)x_3 = 0$
(7) $a_0a_6x_0 + (-a_1a_6 - a_2a_5 - a_3a_4)x_1 + (a_1a_5 - a_2a_6 + a_3a_7)x_2 + (a_1a_4 - a_2a_7 - a_3a_6)x_3 = 0$
(8) $a_0a_7x_0 + (-a_1a_7 + a_2a_4 - a_3a_5)x_1 + (-a_1a_4 - a_2a_7 - a_3a_6)x_2 + (a_1a_5 + a_2a_6 - a_3a_7)x_3 = 0$

Now, build $a_4 * (5) + a_5 * (6) + a_6 * (7) + a_7 * (8)$. Then, we get:

$(a_0x_0 - a_1x_1 - a_2x_2 - a_3x_3)(a_4^2 + a_5^2 + a_6^2 + a_7^2) = 0$. Since the second term cannot be zero (otherwise: $\boldsymbol{a} \in \mathbb{H}$, and then $\boldsymbol{axa} = \boldsymbol{0}$ yields $\boldsymbol{a} = 0$ or $\boldsymbol{x} = \boldsymbol{0}$, a contradiction), we conclude: $a_0x_0 - a_1x_1 - a_2x_2 - a_3x_3 = 0 \succ a_0x_0 = a_1x_1 + a_2x_2 + a_3x_3$ and (1) yields:

$(a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2)x_0 = 0 \succ x_0 = 0$. In a similar way we get from (2): $x_1 = 0$, from (3): $x_2 = 0$ and from (4): $x_3 = 0$, a contradiction.

What about $\boldsymbol{axa}^*$?

If for $\boldsymbol{a} \in \mathbb{O}_1 \cup \mathbb{O}_2 \cup \mathbb{O}_3 \subset \mathbb{S}$ and $\boldsymbol{x} \in \mathbb{H} \subset \mathbb{S}$, $\boldsymbol{a}, \boldsymbol{x} \neq \boldsymbol{0}$, then $(\boldsymbol{ax})\boldsymbol{a}^* \neq \boldsymbol{0}$ *and* $\boldsymbol{a}(\boldsymbol{xa}^*) \neq \boldsymbol{0}$.

Proof:

Let $\boldsymbol{a} \in \mathbb{O}_1 \cup \mathbb{O}_2$ and inspect $(\boldsymbol{ax})\boldsymbol{a}^*$. Suppose the product would be zero. Then:

$\boldsymbol{a}^*\big((\boldsymbol{ax})\boldsymbol{a}^*\big) = (\boldsymbol{a}^*\boldsymbol{a})(\boldsymbol{xa}) = \boldsymbol{0}$ due to Moufang identities, valid in $\mathbb{O}_1$ and $\mathbb{O}_2$. But, since $\boldsymbol{a}^*\boldsymbol{a}$ is the square of the Euclidian norm of $\boldsymbol{a}$ and $\boldsymbol{xa} \neq \boldsymbol{0}$ this leads to a contradiction.

Now, let us inspect $\boldsymbol{a}(\boldsymbol{xa}^*)$. Again, suppose the product would be zero. Then due to flexibility and Moufang identities we get:

$\big(\boldsymbol{a}(\boldsymbol{xa}^*)\big)\boldsymbol{a} = \boldsymbol{a}\big((\boldsymbol{xa}^*)\boldsymbol{a}\big) = (\boldsymbol{ax})(\boldsymbol{a}^*\boldsymbol{a}) = \boldsymbol{0}$. Again, this leads to a contradiction.

Let $\boldsymbol{a} \in \mathbb{O}_3$ and let us inspect $(\boldsymbol{ax})\boldsymbol{a}^*$. Suppose the product would be zero. Then after a voluminous calculation we get the following equations:

(1) $(a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2)x_0 = 0$
(2) $(a_0^2 + a_1^2 - a_2^2 - a_3^2 - a_4^2 - a_5^2 - a_6^2 - a_7^2)x_1 + 2(a_1a_2 - a_0a_3)x_2 + 2(a_0a_2 + a_1a_3)x_3 = 0$
(3) $(a_0^2 - a_1^2 + a_2^2 - a_3^2 - a_4^2 - a_5^2 - a_6^2 - a_7^2)x_2 + 2(a_1a_2 + a_0a_3)x_1 + 2(-a_0a_1 + a_2a_3)x_3 = 0$
(4) $(a_0^2 - a_1^2 - a_2^2 + a_3^2 - a_4^2 - a_5^2 - a_6^2 - a_7^2)x_3 + 2(a_1a_3 - a_0a_2)x_1 + 2(a_0a_1 + a_2a_3)x_3 = 0$
(5) $(a_1a_4 - a_0a_5 - a_3a_6 + a_2a_7)x_1 + (a_2a_4 - a_0a_6 + a_3a_5 - a_1a_7)x_2 + (a_3a_4 - a_0a_7 - a_2a_5 + a_1a_6)x_3 = 0$
(6) $(a_1a_5 + a_0a_4 - a_3a_7 - a_2a_6)x_1 + (a_2a_5 - a_0a_7 - a_3a_4 + a_1a_6)x_2 + (a_3a_5 + a_0a_6 + a_2a_4 + a_1a_7)x_3 = 0$

(7) $(a_1a_6 + a_0a_7 + a_3a_4 + a_2a_5)x_1 + (a_2a_6 + a_0a_4 - a_3a_7 - a_1a_5)x_2 + (a_3a_6 - a_0a_5 + a_2a_7 - a_1a_4)x_3 = 0$

(8) $(a_1a_7 - a_0a_6 + a_3a_5 - a_2a_4)x_1 + (a_2a_7 + a_0a_5 + a_3a_6 + a_1a_4)x_2 + (a_3a_7 + a_0a_4 - a_2a_6 - a_1a_5)x_3 = 0$

Now, build $a_4 *$ (5) + $a_5 *$ (6) + $a_6 *$ (7) + $a_7 *$ (8). This results in:

$(a_1x_1 + a_2x_2 + a_3x_3)({a_4}^2 + {a_5}^2 + {a_6}^2 + {a_7}^2) = 0 \succ a_1x_1 + a_2x_2 + a_3x_3 = 0$ (because otherwise $\boldsymbol{a} \in \mathbb{H}$, which yields a contradiction). This means that vectors $(a_1, a_2, a_3)^{\mathrm{T}}$ and $(x_1, x_2, x_3)^{\mathrm{T}}$ are orthogonal with respect to the ordinary scalar product in 3-space. Moreover, (2) – (4) now transform to:

(2) $({a_0}^2 - {a_1}^2 - {a_2}^2 - {a_3}^2 - {a_4}^2 - {a_5}^2 - {a_6}^2 - {a_7}^2)x_1 + 2a_0(a_2x_3 - a_3x_2) = 0$

(3) $({a_0}^2 - {a_1}^2 - {a_2}^2 - {a_3}^2 - {a_4}^2 - {a_5}^2 - {a_6}^2 - {a_7}^2)x_2 + 2a_0(a_3x_1 - a_1x_3) = 0$

(4) $({a_0}^2 - {a_1}^2 - {a_2}^2 - {a_3}^2 - {a_4}^2 - {a_5}^2 - {a_6}^2 - {a_7}^2)x_3 + 2a_0(a_1x_2 - a_2x_1) = 0$

Abbreviating $r^2 \coloneqq {a_1}^2 + {a_2}^2 + {a_3}^2 + {a_4}^2 + {a_5}^2 + {a_6}^2 + {a_7}^2 \in \mathbb{R}_+, \boldsymbol{a}' \coloneqq (a_1, a_2, a_3)^{\mathrm{T}}$ and $\boldsymbol{x}' \coloneqq (x_1, x_2, x_3)^{\mathrm{T}}$ we get: $({a_0}^2 - r^2)\boldsymbol{x}' + 2a_0(\boldsymbol{a}' \times \boldsymbol{x}') = \boldsymbol{0}$ with cross product $\times$ in 3-space. Now, a_0 cannot be zero, since then $\boldsymbol{x}' = \boldsymbol{0}$ and together with (1) we would get $\boldsymbol{x} = \boldsymbol{0}$, a contradiction. For $\boldsymbol{a}' \neq \boldsymbol{0}$, $\boldsymbol{a}' \times \boldsymbol{x}'$ would be orthogonal to $\boldsymbol{x}'$, but that is not possible. So, we get: $\boldsymbol{a}' = \boldsymbol{0}$ and ${a_0}^2 - r^2 = 0$. Now we end up for (5)-(8):

(5) $a_0(a_5x_1 + a_6x_2 + a_7x_3) = 0$

(6) $a_0(a_4x_1 - a_7x_2 + a_6x_3) = 0$

(7) $a_0(a_7x_1 + a_4x_2 - a_5x_3) = 0$

(8) $a_0(-a_6x_1 + a_5x_2 + a_4x_3) = 0$

Since $a_0 \neq 0$ the terms in brackets have to be zero. If we build $a_5 *$ (5) + $a_4 *$ (6) + $a_7 *$ (7) - $a_6 *$ (8), we get: $({a_4}^2 + {a_5}^2 + {a_6}^2 + {a_7}^2)x_1 = 0$, hence $x_1 = 0$. In a similar way one gets $x_2 = 0$ and $x_3 = 0$, a contradiction.

Now, $((\boldsymbol{ax})\boldsymbol{a}^*)^* = \boldsymbol{a}(\boldsymbol{x}^*\boldsymbol{a}^*)$. Since $(\boldsymbol{ax})\boldsymbol{a}^* \neq \boldsymbol{0}$ for all $\boldsymbol{x} \in \mathbb{H}$, with $\boldsymbol{x} \neq \boldsymbol{0}$, and $\boldsymbol{x} \neq \boldsymbol{0} \Leftrightarrow \boldsymbol{x}^* \neq \boldsymbol{0}$ it turns out, that $\boldsymbol{a}(\boldsymbol{xa}^*) \neq 0$ too.

Let ${\mathbb{O}_1}^- \coloneqq < \boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{e}_6, \boldsymbol{e}_7 >$, ${\mathbb{O}_2}^- \coloneqq < \boldsymbol{e}_8, \boldsymbol{e}_9, \boldsymbol{e}_{10}, \boldsymbol{e}_{11} >$, ${\mathbb{O}_3}^- \coloneqq < \boldsymbol{e}_{12}, \boldsymbol{e}_{13}, \boldsymbol{e}_{14}, \boldsymbol{e}_{15} >$ according to the multiplication table of the sedenions $\mathbb{S}$, where $< \boldsymbol{x}, \boldsymbol{y}, .., \boldsymbol{z} >$ is the vector space generated by $\boldsymbol{x}, \boldsymbol{y}, .., \boldsymbol{z}$. Then we get:

If $\boldsymbol{a} \in {\mathbb{O}_1}^- \cup {\mathbb{O}_2}^- \cup {\mathbb{O}_3}^- \subset \mathbb{S}$ with $\boldsymbol{a}^2 = -\boldsymbol{1}$ and $\boldsymbol{x} \in \mathbb{H} \subset \mathbb{S}$, then $(\boldsymbol{ax})\boldsymbol{a}^* = \boldsymbol{a}(\boldsymbol{xa}^*) = \boldsymbol{x}^*$.

Proof:

Due to the Cayley-Dickson doubling process in construction of the octonions $\mathbb{O} = \mathbb{O}_1$ on base of the quaternions $\mathbb{H}$, we can write $\boldsymbol{a} \in {\mathbb{O}_1}^-$ as $\boldsymbol{a} = (\boldsymbol{0}, \boldsymbol{b})$ with $\boldsymbol{b} \in \mathbb{H}$ and identify $\boldsymbol{x}$ with $(\boldsymbol{x}, \boldsymbol{0})$. Then $\boldsymbol{a}^2 = (\boldsymbol{0}, \boldsymbol{b})^2 = (\boldsymbol{b}^2, \boldsymbol{0}) = (-\boldsymbol{1}, \boldsymbol{0})$, hence $\boldsymbol{b}^2 = -\boldsymbol{1}$, so: $(\boldsymbol{ax})\boldsymbol{a}^* = \big((\boldsymbol{0}, \boldsymbol{b}) * (\boldsymbol{x}, \boldsymbol{0})\big) * (\boldsymbol{0}, -\boldsymbol{b}) = (\boldsymbol{0}, \boldsymbol{bx}^*) * (\boldsymbol{0}, -\boldsymbol{b}) = (-\boldsymbol{b}(\boldsymbol{bx}^*), \boldsymbol{0}) \doteq (-\boldsymbol{b}^2\boldsymbol{x}^*, \boldsymbol{0}) = (\boldsymbol{x}^*, \boldsymbol{0}) = \boldsymbol{x}^*$ where equality $\doteq$ is valid due to associative identity within $\mathbb{H}$.

Due to the Cayley-Dickson doubling process in construction of the sedenions on base of the octonions $\mathbb{O} = \mathbb{O}_1$, we can write $\boldsymbol{a} \in {\mathbb{O}_2}^- \cup {\mathbb{O}_3}^- \subset \mathbb{S}$ as $\boldsymbol{a} = (\boldsymbol{0}, \boldsymbol{b})$ with $\boldsymbol{b} \in \mathbb{O}$ and identify $\boldsymbol{x} \in \mathbb{H}$ with $(\boldsymbol{x}, \boldsymbol{0})$. Then $\boldsymbol{b}^2 = -\boldsymbol{1}$ like before, and: $(\boldsymbol{ax})\boldsymbol{a}^* = \big((\boldsymbol{0}, \boldsymbol{b}) * (\boldsymbol{x}, \boldsymbol{0})\big) * (\boldsymbol{0}, -\boldsymbol{b}) = (\boldsymbol{0}, \boldsymbol{bx}^*) * (\boldsymbol{0}, -\boldsymbol{b}) = (-\boldsymbol{b}(\boldsymbol{bx}^*), \boldsymbol{0}) \doteq (-\boldsymbol{b}^2\boldsymbol{x}^*, \boldsymbol{0}) = (\boldsymbol{x}^*, \boldsymbol{0}) = \boldsymbol{x}^*$ where equality $\doteq$ is valid due to alternative identity within $\mathbb{O}$.

Let us come to a very special result, concerning summed roots of unity in $\mathbb{O}_1 \cup \mathbb{O}_2 \cup \mathbb{O}_3$.

For $\boldsymbol{x} \in \mathbb{H} \subset \mathbb{S}$ and $\boldsymbol{a} \in \mathbb{O}_1 \cup \mathbb{O}_2 \cup \mathbb{O}_3 \subset \mathbb{S}$ with $(\boldsymbol{a} \coloneqq a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 + a_4\boldsymbol{e}_4$ with ${a_1}^2 + {a_2}^2 + {a_3}^2 + {a_4}^2 = 1$ or $\boldsymbol{a} \coloneqq a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 - a_1\boldsymbol{e}_5 - a_2\boldsymbol{e}_6 - a_3\boldsymbol{e}_7$ with $2({a_1}^2 + {a_2}^2 + {a_3}^2) = 1)$ [10] and $((x_1, x_2, x_3) = \lambda(a_1, a_2, a_3),\ \lambda \in \mathbb{R}$, $if\ \boldsymbol{a} \in \mathbb{O}_3)$, we have:
$\boldsymbol{axa} = -\boldsymbol{x}^* - 2(a_1x_1 + a_2x_2 + a_3x_3)\boldsymbol{a}$.

Proof:

(1) $\boldsymbol{a} \coloneqq a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 + a_4\boldsymbol{e}_4$ and $\boldsymbol{a} \in \mathbb{O}_1 \cup \mathbb{O}_2$:
Due to Flexibility in $\mathbb{O}_1 \cup \mathbb{O}_2$ we may omit brackets in $\boldsymbol{axa}$. So, we get: $\boldsymbol{axa} =$
$((a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 + a_4\boldsymbol{e}_4)(x_0\boldsymbol{e}_0 + x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3))(a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 + a_4\boldsymbol{e}_4) =$
$((-a_1x_1 - a_2x_2 - a_3x_3)\boldsymbol{e}_0 + (a_1x_0 + a_2x_3 - a_3x_2)\boldsymbol{e}_1 + (a_2x_0 + a_3x_1 - a_1x_3)\boldsymbol{e}_2 + (a_3x_0 + a_1x_2 - a_2x_1)\boldsymbol{e}_3 + a_4x_0\boldsymbol{e}_4 - a_4x_1\boldsymbol{e}_5 - a_4x_2\boldsymbol{e}_6 - a_4x_3\boldsymbol{e}_7)(a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 + a_4\boldsymbol{e}_4) = \boldsymbol{\ldots} =$
$-({a_1}^2 + {a_2}^2 + {a_3}^2 + {a_4}^2)x_0\boldsymbol{e}_0 + ((-{a_1}^2 + {a_2}^2 + {a_3}^2 + {a_4}^2)x_1 - 2a_1(a_2x_2 + a_3x_3))\boldsymbol{e}_1 + (({a_1}^2 - {a_2}^2 + {a_3}^2 + {a_4}^2)x_2 - 2a_2(a_1x_1 + a_3x_3))\boldsymbol{e}_2 + (({a_1}^2 + {a_2}^2 - {a_3}^2 + {a_4}^2)x_3 - 2a_3(a_1x_1 + a_2x_2))\boldsymbol{e}_3 - 2a_4(a_1x_1 + a_2x_2 + a_3x_3)\boldsymbol{e}_4 =$
$-({a_1}^2 + {a_2}^2 + {a_3}^2 + {a_4}^2)x_0\boldsymbol{e}_0 + (({a_1}^2 + {a_2}^2 + {a_3}^2 + {a_4}^2)x_1 - 2a_1(a_1x_1 + a_2x_2 + a_3x_3))\boldsymbol{e}_1 + (({a_1}^2 + {a_2}^2 + {a_3}^2 + {a_4}^2)x_2 - 2a_2(a_1x_1 + a_2x_2 + a_3x_3))\boldsymbol{e}_2 + (({a_1}^2 + {a_2}^2 + {a_3}^2 + {a_4}^2)x_3 - 2a_3(a_1x_1 + a_2x_2 + a_3x_3))\boldsymbol{e}_3 - 2a_4(a_1x_1 + a_2x_2 + a_3x_3)\boldsymbol{e}_4 =$
$-({a_1}^2 + {a_2}^2 + {a_3}^2 + {a_4}^2)\boldsymbol{x}^* - 2(a_1x_1 + a_2x_2 + a_3x_3)\boldsymbol{a} =$
$-\boldsymbol{x}^* - 2(a_1x_1 + a_2x_2 + a_3x_3)\boldsymbol{a}$

(2) $\boldsymbol{a} \coloneqq a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 + a_4\boldsymbol{e}_4$ and $\boldsymbol{a} \in \mathbb{O}_3$:
Due to Flexibility in $\mathbb{O}_3$ we may omit brackets in $\boldsymbol{axa}$. Using multiplication table in A1, we get: $\boldsymbol{axa} =$
$((a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 + a_4\boldsymbol{e}_4)(x_0\boldsymbol{e}_0 + x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3))(a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 + a_4\boldsymbol{e}_4) =$
$((-a_1x_1 - a_2x_2 - a_3x_3)\boldsymbol{e}_0 + (a_1x_0 + a_2x_3 - a_3x_2)\boldsymbol{e}_1 + (a_2x_0 + a_3x_1 - a_1x_3)\boldsymbol{e}_2 + (a_3x_0 + a_1x_2 - a_2x_1)\boldsymbol{e}_3 + a_4x_0\boldsymbol{e}_4 + a_4x_1\boldsymbol{e}_5 + a_4x_2\boldsymbol{e}_6 + a_4x_3\boldsymbol{e}_7)(a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 + a_4\boldsymbol{e}_4) = \boldsymbol{\ldots} =$
$-({a_1}^2 + {a_2}^2 + {a_3}^2 + {a_4}^2)x_0\boldsymbol{e}_0 + ({a_1}^2 + {a_2}^2 + {a_3}^2 + {a_4}^2)(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3) - 2a_1(a_1x_1 + a_2x_2 + a_3x_3))\boldsymbol{e}_1 - 2a_2(a_1x_1 + a_2x_2 + a_3x_3))\boldsymbol{e}_2 - 2a_3(a_1x_1 + a_2x_2 + a_3x_3))\boldsymbol{e}_3 - 2a_4(a_1x_1 + a_2x_2 + a_3x_3)\boldsymbol{e}_4 -$
$2a_4(a_2x_3 - a_3x_2)\boldsymbol{e}_5 - 2a_4(a_3x_1 - a_1x_3)\boldsymbol{e}_6 - 2a_4(a_1x_2 - a_2x_1)\boldsymbol{e}_7$
The last 3 terms represent $2a_4$-times the cross product of (a_1, a_2, a_3) and (x_1, x_2, x_3) with respect to base elements $\boldsymbol{e}_5, \boldsymbol{e}_6, \boldsymbol{e}_7$. So, the 3 terms are zero, if $(x_1, x_2, x_3) = \lambda(a_1, a_2, a_3)$ with $\lambda \in \mathbb{R}$. And then we end up with:
$\boldsymbol{axa} = -\boldsymbol{x}^* - 2(a_1x_1 + a_2x_2 + a_3x_3)\boldsymbol{a}.$

(3) $\boldsymbol{a} \coloneqq a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 - a_1\boldsymbol{e}_5 - a_2\boldsymbol{e}_6 - a_3\boldsymbol{e}_7$ and $\boldsymbol{a} \in \mathbb{O}_1 \cup \mathbb{O}_2$:
Due to Flexibility in $\mathbb{O}_1 \cup \mathbb{O}_2$ we may omit brackets in $\boldsymbol{axa}$. So, we get: $\boldsymbol{axa} =$
$((a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 - a_1\boldsymbol{e}_5 - a_2\boldsymbol{e}_6 - a_3\boldsymbol{e}_7)(x_0\boldsymbol{e}_0 + x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3))(a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 - a_1\boldsymbol{e}_5 - a_2\boldsymbol{e}_6 - a_3\boldsymbol{e}_7) =$

[10] Indices 8-11 in $\mathbb{O}_2$ and 12-15 in $\mathbb{O}_3$ replaced by 4-7

$((-a_1x_1 - a_2x_2 - a_3x_3)\boldsymbol{e}_0 + (a_1x_0 + a_2x_3 - a_3x_2)\boldsymbol{e}_1 + (a_2x_0 + a_3x_1 - a_1x_3)\boldsymbol{e}_2 + (a_3x_0 + a_1x_2 - a_2x_1)\boldsymbol{e}_3 + (-a_1x_1 - a_2x_2 - a_3x_3)\boldsymbol{e}_4 + (-a_1x_0 + a_2x_3 - a_3x_2)\boldsymbol{e}_5 + (-a_2x_0 + a_3x_1 - a_1x_3)\boldsymbol{e}_6 + (-a_3x_0 + a_1x_2 - a_2x_1)\boldsymbol{e}_7)(a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 - a_1\boldsymbol{e}_5 - a_2\boldsymbol{e}_6 - a_3\boldsymbol{e}_7) = \ldots =$
$-2({a_1}^2 + {a_2}^2 + {a_3}^2)x_0\boldsymbol{e}_0 + 2(({a_2}^2 + {a_3}^2)x_1 - a_1(a_2x_2 + a_3x_3))\boldsymbol{e}_1 + 2(({a_1}^2 + {a_3}^2)x_2 - a_2(a_1x_1 + a_3x_3))\boldsymbol{e}_2 + 2(({a_1}^2 + {a_2}^2)x_3 - a_3(a_1x_1 + a_2x_2))\boldsymbol{e}_3 +$
$2a_1(a_1x_1 + a_2x_2 + a_3x_3)\boldsymbol{e}_5 + 2a_2(a_1x_1 + a_2x_2 + a_3x_3)\boldsymbol{e}_6 + 2a_3(a_1x_1 + a_2x_2 + a_3x_3)\boldsymbol{e}_7 =$
$-2({a_1}^2 + {a_2}^2 + {a_3}^2)\boldsymbol{x}^* - 2(a_1x_1 + a_2x_2 + a_3x_3)\boldsymbol{a} =$
$-\boldsymbol{x}^* - 2(a_1x_1 + a_2x_2 + a_3x_3)\boldsymbol{a}$

(4) $\boldsymbol{a} \coloneqq a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 - a_1\boldsymbol{e}_5 - a_2\boldsymbol{e}_6 - a_3\boldsymbol{e}_7$ and $\boldsymbol{a} \in \mathbb{O}_3$:
Due to Flexibility in $\mathbb{O}_3$ we may omit brackets in $\boldsymbol{axa}$. Using multiplication table in A1, we get: $\boldsymbol{axa} =$
$((a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 - a_1\boldsymbol{e}_5 - a_2\boldsymbol{e}_6 - a_3\boldsymbol{e}_7)(x_0\boldsymbol{e}_0 + x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3))(a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 - a_1\boldsymbol{e}_5 - a_2\boldsymbol{e}_6 - a_3\boldsymbol{e}_7) =$
$((-a_1x_1 - a_2x_2 - a_3x_3)\boldsymbol{e}_0 + (a_1x_0 + a_2x_3 - a_3x_2)\boldsymbol{e}_1 + (a_2x_0 + a_3x_1 - a_1x_3)\boldsymbol{e}_2 + (a_3x_0 + a_1x_2 - a_2x_1)\boldsymbol{e}_3 + (a_1x_1 + a_2x_2 + a_3x_3)\boldsymbol{e}_4 - (a_1x_0 + a_2x_3 - a_3x_2)\boldsymbol{e}_5 - (a_2x_0 + a_3x_1 - a_1x_3)\boldsymbol{e}_6 - (a_3x_0 + a_1x_2 - a_2x_1)\boldsymbol{e}_7)(a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 - a_1\boldsymbol{e}_5 - a_2\boldsymbol{e}_6 - a_3\boldsymbol{e}_7) =$
$\ldots =$
$-2({a_1}^2 + {a_2}^2 + {a_3}^2)x_0\boldsymbol{e}_0 + 2(({a_2}^2 + {a_3}^2)x_1 - a_1(a_2x_2 + a_3x_3))\boldsymbol{e}_1 + 2(({a_1}^2 + {a_3}^2)x_2 - a_2(a_1x_1 + a_3x_3))\boldsymbol{e}_2 + 2(({a_1}^2 + {a_2}^2)x_3 - a_3(a_1x_1 + a_2x_2))\boldsymbol{e}_3 +$
$2(({a_1}^2 - {a_2}^2 - {a_3}^2)x_1 + 2a_1(a_2x_2 + a_3x_3))\boldsymbol{e}_5 + 2((-{a_1}^2 + {a_2}^2 - {a_3}^2)x_2 + 2a_2(a_1x_1 + a_3x_3))\boldsymbol{e}_6 + 2((-{a_1}^2 - {a_2}^2 + {a_3}^2)x_3 + 2a_3(a_1x_1 + a_2x_2))\boldsymbol{e}_7 =$
$-2({a_1}^2 + {a_2}^2 + {a_3}^2)\boldsymbol{x}^* - 2(a_1x_1 + a_2x_2 + a_3x_3)\boldsymbol{a} - 2({a_1}^2 + {a_2}^2 + {a_3}^2)(x_1\boldsymbol{e}_5 + x_2\boldsymbol{e}_6 + x_3\boldsymbol{e}_7) + 2(a_1x_1 + a_2x_2 + a_3x_3)(a_1\boldsymbol{e}_5 + a_2\boldsymbol{e}_6 + a_3\boldsymbol{e}_7) =$
$-\boldsymbol{x}^* - 2(a_1x_1 + a_2x_2 + a_3x_3)\boldsymbol{a} - (x_1\boldsymbol{e}_5 + x_2\boldsymbol{e}_6 + x_3\boldsymbol{e}_7) + 2(a_1x_1 + a_2x_2 + a_3x_3)(a_1\boldsymbol{e}_5 + a_2\boldsymbol{e}_6 + a_3\boldsymbol{e}_7)$
The grey marked term is zero, if $(x_1, x_2, x_3) = \lambda(a_1, a_2, a_3),\ \lambda \in \mathbb{R}$.

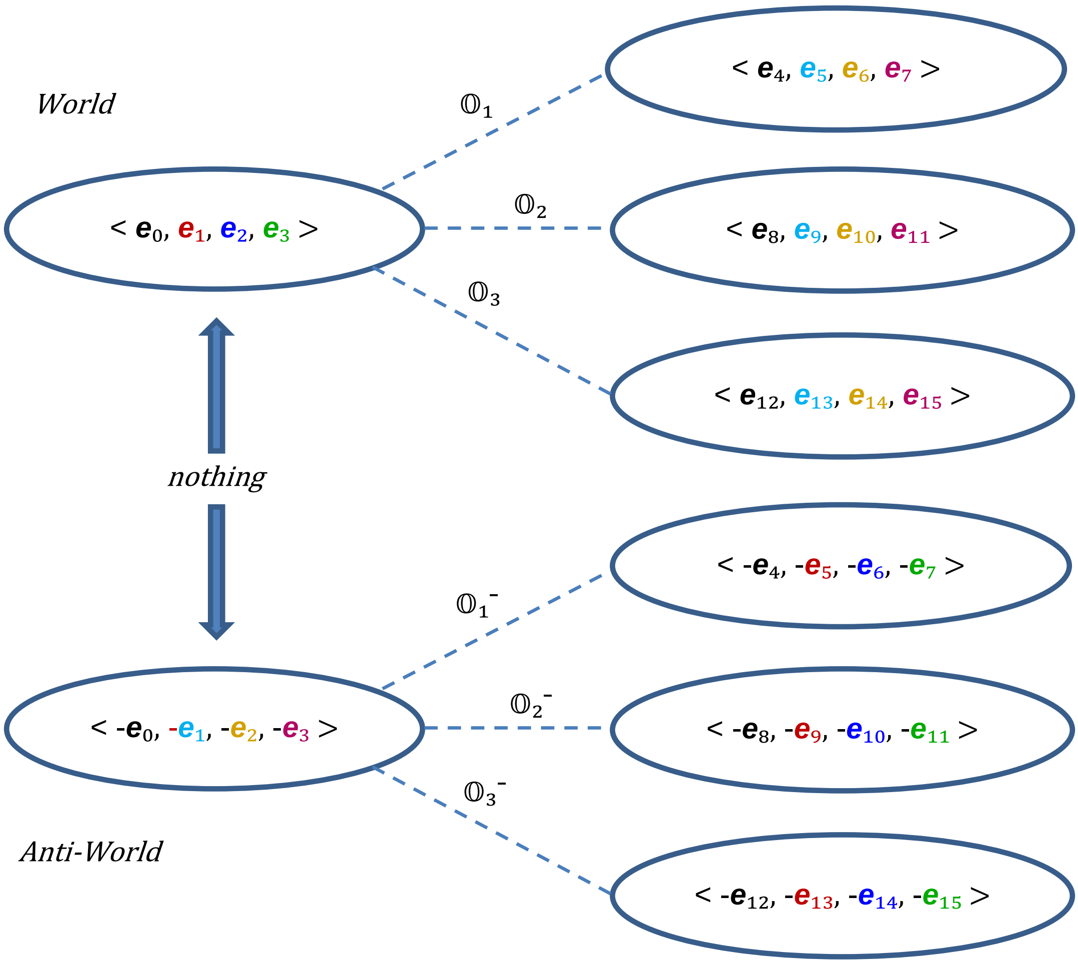
From appendix A1 we can derive the following additional result:

For $\boldsymbol{a} \in \mathbb{O}_1 \cup \mathbb{O}_2 \cup \mathbb{O}_3 \subset \mathbb{S}$ and $\boldsymbol{x} \in \mathbb{H} \subset \mathbb{S}$, then $\|\boldsymbol{ax}\|^2 = \|\boldsymbol{xa}\|^2 = \|\boldsymbol{a}\|^2\|\boldsymbol{x}\|^2$.

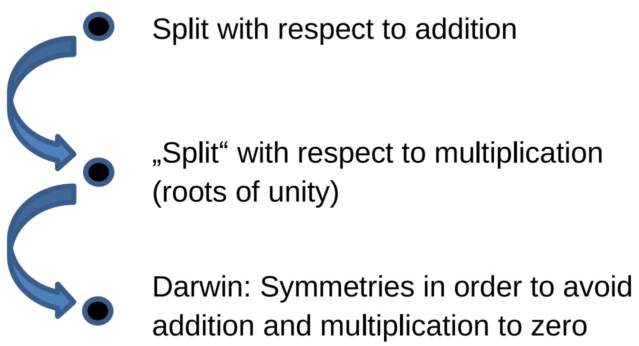
Here $\|\boldsymbol{a}\|^2 = \boldsymbol{aa}^* \in \mathbb{R}_+$. For $\boldsymbol{a} \in \mathbb{O}_1 \cup \mathbb{O}_2$ this is clear, since both subalgebras are isomorphic to the octonions and therefore are normed algebras. For $\mathbb{O}_3$ this means, that compatibility of the multiplication with the norm is partially valid.

B A view of nothing

The theory, presented here, states, that our universe is just curved spacetime and the only objects inside are made of black holes (with respect to curvature, not just gravity). So, elementary particles are black holes too. Moreover, our universe itself is assumed to be a black hole. And as such, it must be embedded in a larger "universe". Its quasi-periodical lifecycle and the interaction within this enclosing universe is partly handled in [2]. The split of black holes of type "elementary particles" within our universe is shown in the next picture.



So, I assume a certain process in creation of our world, illustrated in the following picture:



Is the theory nihilistic? I do not think so. Since, if it would be true, its truth would be part of nothing. And so, irrelevant for all life, especially for us.

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